Perturbation Methods

GM01

Dr. Helen J. Wilson

Autumn Term 2008

1 Introduction

1.1 What are perturbation methods?

Many physical processes are described by equations which cannot be solved analytically. Working in mathematical modelling, you would have to be exceptionally lucky never to have this happen to you!

There are two main approaches to dealing with these equations:

- *numerical methods* and
- *analytic approximations*.

Numerical methods will be taught in GM04; here we focus on analytical approximations. This series of lectures is a very brief introduction to how to systematically construct an approximation of the solution to a problem that is otherwise untractable.

The methods all rely on there being a parameter in the problem that is relatively small: $\varepsilon \ll 1$. The most common example you may have seen before[1] is that of high-Reynolds number fluid mechanics, in which a viscous boundary layer is found close to a solid surface. Note that in this case the standard physical parameter Re is large: our small parameter is $\varepsilon = Re^{-1}$.

1.2 Why use perturbation methods?

There are two major types of use for these methods. The first is in modelling physical applications which, like high-Reynolds number flow, naturally supply such a small parameter. This kind of application is fairly common, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics.

The second use of perturbation methods is coupled with numerical methods. Although computed solutions to a problem can be very accurate, and available for very complex systems, there are two major drawbacks to numerical computation: and perturbation methods can help with both of these.

There is always a concern with numerical calculations about whether the code is correct. A helpful check can be to push one or more of the physical parameters of the problem to extreme values and compare the numerical results with a perturbation solution worked out when that parameter is small (or large).

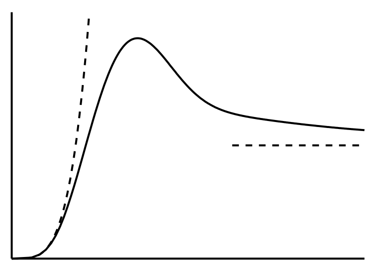
There are other ways of checking code, however; more importantly, a numerical calculation does not often provide insight into the underlying physics. Sometimes (surprisingly often in practice) the simplified problems presented by taking a limiting case have a simplified physics which nonetheless encapsulates some of the key mechanisms from the full problem – and these mechanisms can then be better understood through perturbation methods.

[1] Don't worry if this is new to you. It's an application area: the theory and methods do not depend on it.

1.3 A real research example

This comes from my own research[2]. I will not present the equations or the working here: but the problem in question is the stability of a polymer extrusion flow. The parameter varied is wavelength: and for both very long waves (wavenumber $k \ll 1$) and very short waves ($k^{-1} \ll 1$) the system is much simplified. The long-wave case, in particular, gives very good insight into the physics of the problem.

If we look at the plot of growth rate of the instability against wavenumber (inverse wavelength):



we can see good agreement between the perturbation method solutions (the dotted lines) and the numerical calculations (solid curve): this kind of agreement gives confidence in the numerics in the middle region where perturbation methods can't help.

1.4 References

The principal reference for this course is the book by Hinch. In particular, many of the examples and exercises are taken from it. The others are also very good – your choice is really a question of style preference. Also see `http://www.ucl.ac.uk/~ucahhwi/GM01/`.

- Hinch, *Perturbation methods*
- Van Dyke, *Perturbation methods in fluid mechanics*
- Kevorkian & Cole, *Perturbation methods in applied mathematics*
- Bender & Orszag, *Advanced mathematical methods for scientists and engineers*

1.5 How I teach

This subject is very much about **methods**: as such, it is learned by doing. I will hand out exercises at appropriate moments – it's very much in your interest to work through these in your own time. When time allows, I'll also get you to try some small problems during lectures. Please ask if anything isn't clear or you need a bit more help. If you haven't understood then I probably need to go through it again for everyone else too; so please speak out! I'm also happy for you to come to my office to ask questions. My office hours are Wednesday 11-12 and Friday 12-1, but feel free to try at other times as well.

[2] H J Wilson & J M Rallison. *Journal of Non-Newtonian Fluid Mechanics*, **72**, 237–251, (1997)

2 Regular perturbation expansions

2.1 Example algebraic equation

Let us look first at the equation

$$x^2 + \varepsilon x - 1 = 0. \tag{1}$$

This is a quadratic so of course we can solve it, regardless of ε:

$$x = -\frac{1}{2}\varepsilon \pm \sqrt{\left(1 + \frac{1}{4}\varepsilon^2\right)}.$$

If ε is small, $\varepsilon \ll 1$, we can then expand for small ε (using the Taylor series for the square root[3]) to have

$$x = \begin{cases} 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \cdots \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{128}\varepsilon^4 + \cdots \end{cases}$$

2.2 Expanding directly from the equation

We were lucky with the equation above that we could solve the whole thing analytically. Here we will try something which doesn't depend quite so heavily on such luck.

The first step is to look at the problem for $\varepsilon = 0$. We are expecting ε to be small, so the answer to this question should[4] be close to the true answer to the problem.

Setting $\varepsilon = 0$ gives

$$x^2 - 1 = 0$$

with roots $x = 1$ and $x = -1$. Let us look at the root near $x = -1$, and try the expansion

$$x = -1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

[3]Actually, in this case the expansion is valid for $|\varepsilon| < 2$; but we will usually require $\varepsilon \ll 1$

[4]This is a massive assumption. If it is true, then a regular perturbation expansion will do the job: if not, we need a singular expansion. We'll see them soon enough.

If we substitute this into equation (1) we get:

$$\begin{array}{cccccccccccccc}
1 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + & \varepsilon^2 x_1^2 & - & 2\varepsilon^3 x_3 & + & 2\varepsilon^3 x_1 x_2 & + & \cdots & \\
 & - & \varepsilon & + & \varepsilon^2 x_1 & & & + & \varepsilon^3 x_2 & & & + & \cdots & \\
-1 & & & & & & & & & & & & & = 0
\end{array}$$

We equate powers of ε:

$$\begin{array}{clrlcl}
\varepsilon^0 & : & 1 - 1 & = 0 & & \\
\varepsilon^1 & : & -2x_1 - 1 & = 0 & , & x_1 = -\frac{1}{2} \\
\varepsilon^2 & : & -2x_2 + x_1^2 + x_1 & = 0 & , & x_2 = -\frac{1}{8} \\
\varepsilon^3 & : & -2x_3 + 2x_1 x_2 + x_2 & = 0 & , & x_3 = 0
\end{array}$$

Note that the equation at ε^0 was automatically satisfied. This will always happen if we have solved the $\varepsilon = 0$ equation correctly.

The series we have found is

$$x = -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^4).$$

This matches with the exact solution. Remember the notation $O(\varepsilon^4)$: this means that the missing terms from the equation tend to zero *at least as fast as* ε^4, as $\varepsilon \to 0$.

2.3 Example differential equation

Suppose we are trying to solve the following differential equation in $x \geq 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) - \varepsilon f^2(x) = 0, \qquad f(0) = 2. \tag{2}$$

We can't solve this directly. However, let's look at $\varepsilon = 0$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} + f(x) = 0, \qquad f(0) = 2,$$

which has solution

$$f(x) = 2e^{-x}.$$

So in the same way as for the algebraic equation, let us try a regular perturbation expansion in ε:

$$f = 2e^{-x} + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \varepsilon^3 f_3(x) + \cdots$$

where in order to satisfy the initial condition $f(0) = 2$, we will have $f_1(0) = f_2(0) = f_3(0) = \cdots = 0$. Substituting into (2) gives

$$\begin{array}{cccccccccccc}
-2e^{-x} & + & \varepsilon f_1'(x) & + & \varepsilon^2 f_2'(x) & + & \varepsilon^3 f_3'(x) & & & +\cdots & \\
+2e^{-x} & + & \varepsilon f_1(x) & + & \varepsilon^2 f_2(x) & + & \varepsilon^3 f_3(x) & & & +\cdots & \\
 & - & 4\varepsilon e^{-2x} & - & 4\varepsilon^2 e^{-x} f_1(x) & - & \varepsilon^3 f_1^2(x) & - & 4\varepsilon^3 e^{-x} f_2(x) & +\cdots & = 0
\end{array}$$

and we can collect powers of ε:

$$\begin{array}{rcrl}
\varepsilon^0 & : & -2e^{-x} + 2e^{-x} & = 0 \\
\varepsilon^1 & : & f_1'(x) + f_1(x) - 4e^{-2x} & = 0 \\
\varepsilon^2 & : & f_2'(x) + f_2(x) - 4e^{-x} f_1(x) & = 0 \\
\varepsilon^3 & : & f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) & = 0
\end{array}$$

The order ε^0 (or 1) equation is satisfied automatically.

Order ε terms. The equation

$$f_1'(x) + f_1(x) = 4e^{-2x}$$

has solution

$$f_1(x) = -4e^{-2x} + c_1 e^{-x}$$

and the boundary condition $f_1(0) = 0$ gives $c_1 = 4$:

$$f_1(x) = 4(e^{-x} - e^{-2x}).$$

Order ε^2 terms. The equation becomes

$$f_2'(x) + f_2(x) = 4e^{-x} f_1(x) \implies f_2'(x) + f_2(x) = 16e^{-x}(e^{-x} - e^{-2x})$$

with solution

$$f_2(x) = 8(-2e^{-2x} + e^{-3x}) + c_2 e^{-x}$$

and the boundary condition $f_2(0) = 0$ gives $c_2 = 8$:

$$f_2(x) = 8(e^{-x} - 2e^{-2x} + e^{-3x}).$$

Order ε^3 terms. The equation is

$$f_3'(x) + f_3(x) - f_1^2(x) - 4e^{-x} f_2(x) = 0$$

which becomes

$$\begin{aligned}
f_3'(x) + f_3(x) &= [4(e^{-x} - e^{-2x})]^2 + 4e^{-x}[8(e^{-x} - 2e^{-2x} + e^{-3x})] \\
&= 48(e^{-2x} - 2e^{-3x} + e^{-4x})
\end{aligned}$$

The solution to this equation is

$$f_3(x) = 16(-3e^{-2x} + 3e^{-3x} - e^{-4x}) + c_3 e^{-x}.$$

Applying the boundary condition $f_3(0) = 0$ gives $c_3 = 16$ so

$$f_3(x) = 16(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}).$$

The solution we have found is:

$$\begin{aligned}
f(x) = 2e^{-x} + 4\varepsilon(e^{-x} - e^{-2x}) + 8\varepsilon^2(e^{-x} - 2e^{-2x} + e^{-3x}) \\
+ 16\varepsilon^3(e^{-x} - 3e^{-2x} + 3e^{-3x} - e^{-4x}) + \cdots
\end{aligned} \tag{3}$$

This is an example of a case where carrying out a perturbation expansion can give us an insight into the full solution. Notice that, for the terms we have calculated,

$$f_n(x) = 2^{n+1}e^{-x}(1-e^{-x})^n,$$

suggesting a *guessed* full solution

$$f(x) = \sum_{n=0}^{\infty} \varepsilon^n 2^{n+1} e^{-x}(1-e^{-x})^n = 2e^{-x}\sum_{n=0}^{\infty}[2\varepsilon(1-e^{-x})]^n = \frac{2e^{-x}}{1-2\varepsilon(1-e^{-x})}.$$

We can check this solution, and it is indeed the correct solution to the ODE of equation (2).

Exercise 1: Find the first three terms of an expansion for each root of the following equation:

$$x^3 - (2-\varepsilon)x^2 - x + 2 + \varepsilon = 0.$$

Exercise 2: Try a regular perturbation expansion in the following differential equation:

$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1, \qquad y(0) = 0, \qquad y(\pi/2) = 0.$$

Calculate the first three terms, that is, up to order ε^2. Apply the boundary conditions at each order.

2.4 Warning signs

So far we have looked at expansions which work. In the sections to come we will see a variety of ways this straightforward method can fail.

As a preview, if you start out using the naïve method, these are a few possible warning signs that something else is going on.

One of the powers of ε produces an insoluble equation

By this I don't mean a differential equation with no analytic solution: that is just bad luck. Rather I mean an equation of the form $x_1 + 1 - x_1 = 0$ which cannot be satisfied by any value of x_1.

The equation at $\varepsilon = 0$ doesn't give the right number of solutions

A polynomial of degree n should have n roots (not necessarily all distinct); a linear ODE of order n should have n solutions. If the equation produced by setting $\varepsilon = 0$ has less solutions then this method will not give all the possible solutions to the full equation. This happens when the coefficient of the highest power (for a polynomial) or the highest derivative (for a differential equation) is zero when $\varepsilon = 0$.

The coefficients of ε can grow without bound

In the case of an expansion $f(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$, the series may not be valid for some values of x if some or all of the $f_i(x)$ become very large. Say, for example, that $f_2(x) \to \infty$ while $f_1(x)$ remains finite. Then $\varepsilon f_1(x)$ is no longer strictly larger than $\varepsilon^2 f_2(x)$ and there is serious trouble.

3 Rescaling

3.1 Example algebraic equation

Here our model equation is

$$\varepsilon x^2 + x - 1 = 0. \tag{4}$$

We won't solve it directly just yet; instead let's see what happens if we try our simple expansion method.

We set $\varepsilon = 0$ to have

$$x - 1 = 0,$$

with just the one solution $x = 1$. Since we started with a second-degree polynomial we know there is trouble ahead: but we *can* find the root near $x = 1$ without any difficulty.

Exercise 3: Calculate the root near $x = 1$, up to and including terms of order ε^3.

Now let us look at the true solution to see what's gone wrong.

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}$$

As $\varepsilon \to 0$, the leading-order terms of the two roots are

$$x = 1 + O(\varepsilon); \qquad \text{and} \qquad -\frac{1}{\varepsilon} + O(1).$$

The first of these is amenable to the simplistic approach; we haven't seen the second root because it $\to \infty$ as $\varepsilon \to 0$.

For this second root, let us try a series

$$x = x_{-1}\varepsilon^{-1} + x_0 + \varepsilon x_1 + \cdots$$

We substitute it into (4):

$$\begin{array}{rcccccccl} & x_{-1}^2\varepsilon^{-1} & + & 2x_{-1}x_0 & + & \varepsilon(x_0^2 + 2x_{-1}x_1) & + & \cdots & \\ + & x_{-1}\varepsilon^{-1} & + & x_0 & + & \varepsilon x_1 & + & \cdots & \\ & & - & 1 & & & & & = 0 \end{array}$$

and collecting powers of ε gives:

$$\begin{array}{rcrcl}
\varepsilon^{-1} & : & x_{-1}^2 + x_{-1} = 0 & ; & x_{-1} = 0 \; , \; -1 \\
\varepsilon^{0} & : & 2x_{-1}x_0 + x_0 - 1 = 0 & ; & x_0 = 1 \; , \; -1 \\
\varepsilon^{1} & : & x_0^2 + 2x_{-1}x_1 + x_1 = 0 & ; & x_1 = -1 \; , \; 1
\end{array}$$

Note that we can now get the expansions for both of the roots using the same method.

3.2 Finding the scaling

What do we do if we can't use the exact solution to tell us about the first term in the series?

We use a trial scaling δ. We put

$$x = \delta(\varepsilon)X$$

with δ being an unknown function of ε, and X being strictly order 1. We call this $X = \text{ord}(1)$: as $\varepsilon \to 0$, X is neither small nor large.

Let's try it for our equation (4):

$$\varepsilon x^2 + x - 1 = 0.$$

We put in the new form, and then look at the different possible values of δ.

	LHS	$=$	$\varepsilon\delta^2X^2$	$+$	δX	$-$	1	$=$	0	
$\delta \ll 1$	LHS	$=$	small	$+$	small	$-$	1	$=$	0	ℵ
$\delta = 1$	LHS	$=$	small	$+$	X	$-$	1	$=$	0	regular root
$1 \ll \delta \ll \frac{1}{\varepsilon}$	$\frac{\text{LHS}}{\delta}$	$=$	small	$+$	X	$-$	small	$=$	0	ℵ
$\delta = \frac{1}{\varepsilon}$	$\frac{\text{LHS}}{\delta}$	$=$	X^2	$+$	X	$-$	small	$=$	0	singular root
$\delta \gg \frac{1}{\varepsilon}$	$\frac{\text{LHS}}{\varepsilon\delta^2}$	$=$	X^2	$+$	small	$-$	small	$=$	0	ℵ

Note that $X = 0$ cannot be a solution because we require $X = \text{ord}(1)$. The scalings that work (in this case $\delta = 1$ and $\delta = \varepsilon^{-1}$) are called **distinguished scalings**.

Exercise 4: Find the distinguished scalings for the following equation:

$$\varepsilon^3x^3 + x^2 + 2x + \varepsilon = 0.$$

and find the first two nonzero terms in the expansion of each root.

Exercise 5: Find the distinguished scalings for the following equation:

$$\varepsilon x^3 + x^2 + (2 - \varepsilon)x + 1 = 0.$$

and find the first two terms in the expansion of each root. [Hint: you may find an exact root – then there will be no more terms in the expansion.]

3.3 Non-integral powers

Try this algebraic equation:

$$(1-\varepsilon)x^2 - 2x + 1 = 0.$$

Setting $\varepsilon = 0$ gives a double root $x = 1$. Now we try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

Substituting in gives

$$\begin{array}{cccccccc} & 1 & + & 2\varepsilon x_1 & + & \varepsilon^2(x_1^2 + 2x_2) & + & \cdots \\ & & - & \varepsilon & - & 2\varepsilon^2 x_1 & + & \cdots \\ - & 2 & - & 2\varepsilon x_1 & - & 2\varepsilon^2 x_2 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At ε^0, as expected, the equation is automatically satisfied. However, at order ε^1, the equation is

$$2x_1 - 1 - 2x_1 = 0 \qquad 1 = 0$$

which we can never satisfy. Something has gone wrong...

In fact in this case we should have expanded in powers of $\varepsilon^{1/2}$. If we set

$$x = 1 + \varepsilon^{1/2} x_{1/2} + \varepsilon x_1 + \cdots$$

then we get

$$\begin{array}{cccccccc} & 1 & + & 2\varepsilon^{1/2} x_{1/2} & + & \varepsilon(x_{1/2}^2 + 2x_1) & + & \cdots \\ & & & & - & \varepsilon & + & \cdots \\ - & 2 & - & 2\varepsilon^{1/2} x_{1/2} & - & 2\varepsilon x_1 & + & \cdots \\ + & 1 & & & & & = & 0 \end{array}$$

At order ε^0 we are still OK as before; at order $\varepsilon^{1/2}$ we have

$$2x_{1/2} - 2x_{1/2} = 0$$

which is also automatically satisfied. We don't get to determine anything until we go to order ε^1, where we get

$$x_{1/2}^2 + 2x_1 - 1 - 2x_1 = 0 \qquad x_{1/2}^2 - 1 = 0$$

giving two solutions $x_{1/2} = \pm 1$. Both of these are valid and will lead to valid expansions if we continue.

We could have predicted that there would be trouble when we found the double root: near a quadratic zero of a function, a change of order $\varepsilon^{1/2}$ in x is needed to change the function value by ε:

3.4 Choosing the expansion series

In the example above, if we had begun by defining $\delta = \varepsilon^{1/2}$ we would have had a straightforward regular perturbation series in δ. But how do we go about spotting what series to use?

In practice, it is usually worth trying an obvious series like ε, ε^2, ε^3 or, if there is a distinguished scaling with fractional powers, then a power series based on that. But this trial-and-error method, while quick, is not guaranteed to succeed.

In general, for an equation in x, we can pose a series

$$x \sim x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + x_2\delta_2(\varepsilon) + \cdots$$

in which x_i is strictly order 1 as $\varepsilon \to 0$ (i.e. tends neither to zero nor infinity) and the series of functions $\delta_i(\varepsilon)$ has $\delta_0(\varepsilon) \gg \delta_1(\varepsilon) \gg \delta_2(\varepsilon) \cdots$ for $\varepsilon \ll 1$.

Then at each order we look for a distinguished scaling. Let us work through an example:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) - 1 - x + \frac{1}{2}x^2 = -\frac{1}{6}\varepsilon.$$

In this case there is a solution near $x = 0$, which we will investigate.

First let us sort out the trigonometric term, expanding it as a Taylor series about $x = 0$:

$$\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)\right] =$$
$$\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] = \sin x + \cos x = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

The governing equation becomes

$$-\frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6) = -\frac{1}{6}\varepsilon.$$

$$x^3 - \frac{x^4}{4} - \frac{x^5}{20} + O(x^6) = \varepsilon.$$

We pose a series

$$x = x_0\delta_0(\varepsilon) + x_1\delta_1(\varepsilon) + \cdots$$

and substitute it. The leading term on the left hand side is $x_0^3\delta_0^3$, and on the right hand side is ε. So we set $\delta_0 = \varepsilon^{1/3}$ and $x_0 = 1$.

Now we have

$$x = \varepsilon^{1/3} + x_1\delta_1(\varepsilon) + \cdots$$

which we substitute into the governing equation. Remembering that $\delta_1 \ll \varepsilon^{1/3}$ and keeping terms up to order $\varepsilon^{2/3}\delta_1$ and $\varepsilon^{4/3}$ (neglecting only terms which are guaranteed to be smaller than one of these), we have

$$3x_1\varepsilon^{2/3}\delta_1 - \frac{\varepsilon^{4/3}}{4} = 0.$$

To make this work, we need $\delta_1 = \varepsilon^{2/3}$ and then $x_1 = 1/12$.

The first two terms of the solution are:

$$x = \varepsilon^{1/3} + \frac{1}{12}\varepsilon^{2/3} + \cdots$$

Exercise 6: Find the first two terms of all four roots of

$$\varepsilon x^4 - x^2 - x + 2 = 0.$$

3.5 Logarithms

HERE BE DRAGONS

There is a worse case than fractional power of ε: logarithms. These are beyond the scope of this course. If you ever have a problem in which a quantity like $\ln(1/\varepsilon)$ appears to be important, either give up or go to a textbook.

4 Scalings with differential equations

4.1 Stretched coordinates

Consider the first-order linear differential equation

$$\varepsilon \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0.$$

Since it is first order, we expect a single solution to the homogeneous equation. If we try our standard method and set $\varepsilon = 0$ we get $f = 0$ which is clearly not a good first term of an expansion!

We can solve this by standard methods to give:

$$f = A_0 \exp\left[-x/\varepsilon\right].$$

This gives us the clue that what we **should** have done was change to a *stretched variable* $z = x/\varepsilon$.

Let us ignore the full solution and simply make that substitution in our governing equation. Note that $\mathrm{d}f/\mathrm{d}x = \mathrm{d}f/\mathrm{d}z\ \mathrm{d}z/\mathrm{d}x = \varepsilon^{-1}\mathrm{d}f/\mathrm{d}z$.

$$\varepsilon\varepsilon^{-1}\frac{\mathrm{d}f}{\mathrm{d}z} + f = 0 \qquad \frac{\mathrm{d}f}{\mathrm{d}z} + f = 0.$$

Now the two terms balance: that is, they are the same order in ε. Clearly the solution to this equation is now $A_0 \exp\left[-z\right]$ and we have found the result.

This is a general principle. For a polynomial, we look for a distinguished scaling of the quantity we are trying to find. For a differential equation, we look for a stretched version of the independent variable.

The process is very similar to that for a polynomial. We use a trial scaling δ and set

$$x = a + \delta(\varepsilon)X.$$

Then we vary δ, looking for values at which the two largest terms in the scaled equation balance.

Let's work through the process for the following equation:

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} - f = 0.$$

Again, we note that if $x = a + \delta X$ then $\mathrm{d}/\mathrm{d}x = \mathrm{d}/\mathrm{d}X\ \mathrm{d}X/\mathrm{d}x = \delta^{-1}\mathrm{d}/\mathrm{d}X$. We substitute in these scalings, and then look at the different possible values of δ.

	LHS	=	$\varepsilon\delta^{-2}\mathrm{d}^2 f/\mathrm{d}X^2$	+	$\delta^{-1}\mathrm{d}f/\mathrm{d}X$	−	f	=	0	
$\delta \ll \varepsilon$	$\varepsilon^{-1}\delta^2$LHS	=	f''	+	small	−	small	=	0	ℵ
$\delta = \varepsilon$	εLHS	=	f''	+	f'	−	small	=	0	
$\varepsilon \ll \delta \ll 1$	δLHS	=	small	+	f'	−	small	=	0	ℵ
$\delta = 1$	LHS	=	small	+	f'	−	f	=	0	
$\delta \gg 1$	LHS	=	small	+	small	−	f	=	0	ℵ

Here the two distinguished stretches are $\delta = \varepsilon$ and $\delta = 1$.

For $\delta = 1$ we can treat this as a regular perturbation expansion:

Exercise 7: Calculate the first two terms of the solution to

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$

where derivatives are order 1 (i.e. $\delta = 1$).

For $\delta = \varepsilon$ we use our new variable $X = \varepsilon^{-1}(x - a)$ and work with the new governing equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}X^2} + \frac{\mathrm{d}f}{\mathrm{d}X} - \varepsilon f = 0$$

Now we try a regular perturbation expansion:

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

We substitute this in and collect powers of ε:

$$\begin{array}{cccccccccc} & f_{0XX} & + & \varepsilon f_{1XX} & + & \varepsilon^2 f_{2XX} & + & \cdots & & \\ + & f_{0X} & + & \varepsilon f_{1X} & + & \varepsilon^2 f_{2X} & + & \cdots & & \\ & & - & \varepsilon f_0 & - & \varepsilon^2 f_1 & + & \cdots & = & 0 \end{array}$$

We then solve at each order:

$$\begin{array}{lll} \varepsilon^0 & : & f_{0XX} + f_{0X} = 0 \qquad f_0 = A_0 + B_0 e^{-X} \\ \varepsilon^1 & : & f_{1XX} + f_{1X} - f_0 = 0 \quad f_1 = A_0 X - B_0 X e^{-X} + A_1 + B_1 e^{-X} \end{array}$$

and so on. Of course, without boundary conditions to apply, this process spawns large numbers of unknown constants. Rescaling to our original variable completes the process:

$$f(x) \sim A_0 + B_0 e^{-(x-a)/\varepsilon} + \varepsilon \left[\left(\frac{x-a}{\varepsilon} \right) \left(A_0 - B_0 e^{-(x-a)/\varepsilon} \right) + A_1 + B_1 e^{-(x-a)/\varepsilon} \right] + \cdots$$

Note that this expansion is only valid where $X = (x-a)/\varepsilon$ is order 1: that is, for x close to the (unknown) value a.

Exercise 8: Find the distinguished stretches for the following differential equation:

$$\varepsilon^3 \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0$$

Find the leading-order term of each solution.

4.2 Nonlinear differential equations: scaling and stretching

Recall that for a linear differential equation, if f is a solution then so is Cf for any constant C. So if $f(x;\varepsilon)$ is a solution as an asymptotic expansion, then Cf is a valid asymptotic solution **even if C is a function of ε.**

The same is not true of nonlinear differential equations. Suppose we are looking at the equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \varepsilon f(x)\frac{\mathrm{d}f}{\mathrm{d}x} + f^2(x) = 0$$

There are two different types of scaling we can apply: we can scale f, or we can stretch x. To get all valid scalings we need to do both of these at once.

Let us take $f = \varepsilon^{\alpha} F$ where F is strictly ord(1), and $x = a + \varepsilon^{\beta} z$ with z also strictly ord(1). Then a derivative scales like $\mathrm{d}/\mathrm{d}x \sim \varepsilon^{-\beta}\mathrm{d}/\mathrm{d}z$ and we can look at the scalings of all our terms:

$$\begin{array}{ccccccc} \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} & + & \varepsilon f(x)\dfrac{\mathrm{d}f}{\mathrm{d}x} & + & f^2(x) & = & 0 \\ \varepsilon^{\alpha}\varepsilon^{-2\beta} & & \varepsilon\varepsilon^{2\alpha}\varepsilon^{-\beta} & & \varepsilon^{2\alpha} & & \end{array}$$

As always with three terms in the equation, there are three possible balances.

- For terms I and II to balance, we need $\alpha - 2\beta = 2\alpha + 1 - \beta$. This gives $\alpha + \beta + 1 = 0$, so that terms I and II scale as $\varepsilon^{2+3\alpha}$, and term III scales as $\varepsilon^{2\alpha}$. We need the balancing terms to dominate, so we also need $2\alpha > 2 + 3\alpha$ which gives $\alpha < -2$.
- For terms I and III to balance, we need $\alpha - 2\beta = 2\alpha$. This gives $\alpha = -2\beta$, so that terms I and III scale as $\varepsilon^{2\alpha}$ and term II scales as $\varepsilon^{1+5\alpha/2}$. Again, we need the non-balancing term to be smaller than the others, so we need $1 + 5\alpha/2 > 2\alpha$, i.e. $\alpha > -2$.
- Finally, to balance terms II and III, we need $2\alpha - \beta + 1 = 2\alpha$ which gives $\beta = 1$. Then terms II and III scale as $\varepsilon^{2\alpha}$ and term I scales as $\varepsilon^{\alpha-2}$, so to make term I smaller than the others we need $\alpha - 2 > 2\alpha$, giving $\alpha < -2$.

We can plot the lines in the α–β plane where these balances occur, and in the regions between, which term (I, II or III) dominates:

We can see that there is a distinguished scaling $\alpha = -2$, $\beta = 1$ where all three terms balance. If we apply this scaling to have $z = (x - a)/\varepsilon$ and $F = \varepsilon^2 f$ then the governing

ODE for $F(z)$ (after multiplication of the whole equation by ε^4) becomes

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + F\frac{\mathrm{d}F}{\mathrm{d}z} + F^2 = 0.$$

This is very nice: but it may not always be appropriate: the boundary conditions may fix the size of either f or x, in which case the best you can do may be one of the simple balance points (i.e. a point (α, β) lying on one of the lines in the diagram).

Exercise 9: Consider the equation

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$

Find the scalings $f = \varepsilon^\alpha F$ and stretches $x = a + \varepsilon^\beta z$ at which two dominant terms balance, and sketch these balance scalings in the α–β plane. Hence determine the critical values of α and β for which all three terms balance. Give also the possible values of β if we are constrained by the boundary conditions to have $\alpha = 0$.

5 Matching: Boundary Layers

Consider the following equation (rather similar to exercise 7):

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0$$

There are two solutions. One is regular:

$$f = f_0(x) + \varepsilon f_1(x) + \cdots$$

Substituting gives, at order 1,

$$f_0' + f_0 = 0 \implies f_0 = a_0 e^{-x}.$$

At order ε we have

$$f_1' + f_1 + f_0'' = 0 \implies f_1 = [a_1 - a_0 x]e^{-x}.$$

The second solution is singular, and the distinguished scaling (to balance the first two terms) is $\delta = \varepsilon$. We introduce a new variable $z = (x-a)/\varepsilon$ to have

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \frac{\mathrm{d}f}{\mathrm{d}z} + \varepsilon f = 0$$

with solution

$$f = F_0(z) + \varepsilon F_1(z) + \cdots$$

At order 1 we have

$$F_0'' + F_0' = 0 \implies F_0' = -B_0 e^{-z}$$
$$F_0(z) = A_0 + B_0 e^{-z}.$$

At order ε we have

$$F_1'' + F_1' + F_0 = 0 \implies F_1' = -A_0 - B_0 z e^{-z} - B_1 e^{-z}$$

$$F_1 = A_1 - A_0 z + B_0[z e^{-z} + e^{-z}] + B_1 e^{-z}.$$

We now have two possible solutions:

$$\begin{aligned} f(x) &\sim a_0 e^{-x} + \varepsilon[a_1 - a_0 x]e^{-x} + \cdots \\ F(z) &\sim A_0 + B_0 e^{-z} + \varepsilon[A_1 - A_0 z + B_0(z e^{-z} + e^{-z}) + B_1 e^{-z}] + \cdots \end{aligned}$$

Question: Will we ever need to use both of these in the same problem?

Answer: The short answer is yes. This is a second-order differential equation, so we are entitled to demand that the solution satisfies two boundary conditions.

Suppose, with the differential equation above, the boundary conditions are

$$f = e^{-1} \text{ at } x = 1 \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = 0 \text{ at } x = 0.$$

We will start by assuming that the unstretched form will do, and apply the boundary condition at $x = 1$ to it:

$$f(x) \sim a_0 e^{-x} + \varepsilon[a_1 - a_0 x]e^{-x} + \cdots$$

$$e^{-1} = a_0 e^{-1} + \varepsilon[a_1 - a_0]e^{-1} + \cdots$$

which immediately yields the conditions $a_0 = 1$, $a_1 = 1$. If we had continued to higher orders we would be able to find the constants there as well.

Now what about the other boundary condition? We have no more disposable constants so we'd be very lucky if it worked! In fact we have

$$f'(x) = -a_0 e^{-x} + \varepsilon[-a_1 - a_0 + a_0 x]e^{-x} + \cdots$$

so at $x = 0$,

$$f'(0) = -1 - 2\varepsilon + \cdots$$

This is where we have to use the other solution. If we fix $a = 0$ in the scaling for z, then the strained region is near $x = 0$. We can re-express the boundary condition in terms of z:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = 0 \text{ at } z = 0.$$

Now applying this boundary condition to our strained expansion gives:

$$F(z) \sim A_0 + B_0 e^{-z} + \varepsilon[A_1 - A_0 z + B_0(ze^{-z} + e^{-z}) + B_1 e^{-z}] + \cdots$$

$$F'(z) = -B_0 e^{-z} + \varepsilon[-A_0 - B_0 z e^{-z} - B_1 e^{-z}] + \cdots$$

and at $z = 0$,

$$F'(0) = -B_0 + \varepsilon[-A_0 - B_1] + \cdots$$

Imposing $F'(0) = 0$ fixes $B_0 = 0$, $B_1 = -A_0$ but does not determine A_0, B_1 or A_1. The solution which matches the $x = 0$ boundary condition is

$$F(z) \sim A_0 + \varepsilon[A_1 - A_0 z - A_0 e^{-z}] + \cdots$$

We now have two perturbation expansions, one valid at $x = 1$ and for most of our region, the other valid near $x = 0$. We have not determined all our parameters. How will we do this? The answer is **matching**.

5.1 Intermediate variable

Suppose (as in the example above) we have two asymptotic solutions to a given problem.

- One scales normally and satisfies a boundary condition somewhere away from the tricky region: we will call this the **outer** solution.
- The other is expressed in terms of a scaled variable, and is valid in a narrow region, (probably) near the other boundary. We will call this the **inner** solution.

In order to make sure that these two expressions both belong to the same real (physical) solution to the problem, we need to match them.

In the case where the outer solution is

$$f(x) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$$

and the inner

$$F(z) = F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$$

with scalings $z = x/\varepsilon$, we will match the two expressions using an **intermediate variable**. This is a new variable, ξ, intermediate in size between x and z, so that when ξ is order 1, x is small and z is large. We can define it as

$$x = \varepsilon^\alpha \xi \implies z = \varepsilon^{\alpha-1}\xi,$$

for α between 0 and 1. It is best to keep α symbolic[5].

The procedure is to substitute ξ into both $f(x)$ and $F(z)$ and then collect orders of ε and force the two expressions to be equal. This is best seen by revisiting the previous example.

Example continued

We had

$$f(x) = e^{-x} + \varepsilon(1-x)e^{-x} + \cdots$$

and

$$F(z) = A_0 + \varepsilon[A_1 - A_0 z - A_0 e^{-z}] + \cdots$$

with $z = x/\varepsilon$. Defining $x = \varepsilon^\alpha \xi$, we look first at $f(x)$:

$$f(x) = e^{-\varepsilon^\alpha \xi} + \varepsilon(1-\varepsilon^\alpha \xi)e^{-\varepsilon^\alpha \xi} + \cdots$$

Since $\varepsilon^\alpha \ll 1$ we can expand the exponential terms to give

$$f(x) = 1 - \varepsilon^\alpha \xi - \frac{1}{2}\varepsilon^{2\alpha}\xi^2 + \varepsilon - 2\varepsilon^{\alpha+1}\xi + O(\varepsilon^2, \varepsilon^{1+2\alpha}, \varepsilon^{3\alpha})$$

Now we look at $F(z)$. Note that $z = \varepsilon^{\alpha-1}\xi$, which is large.

$$F(z) = A_0 + \varepsilon[A_1 - A_0\varepsilon^{\alpha-1}\xi - A_0 e^{-\varepsilon^{\alpha-1}\xi}] + \cdots$$

Here the exponential terms become very small indeed so we neglect them and have

$$F(z) = A_0 - A_0\varepsilon^\alpha \xi + \varepsilon A_1 + \cdots$$

Comparing terms of the two expansions, at order 1 we have

$$1 = A_0$$

[5]However, occasionally you may find it quicker to pick a value of $\alpha = 1/2$, say. Be warned: sometimes there is only a specific range of α which works.

and at order ε^α,

$$-\xi = -A_0\xi$$

which is automatically satisfied if $A_0 = 1$. If we fix $\alpha > 1/2$ then the next term is order ε, giving

$$1 = A_1.$$

The next term in the outer expansion is order $\varepsilon^{2\alpha}$, but to match that we would have to go to order ε^2 in the inner expansion.

We have now determined all the constants to this order: so in the outer we have

$$f(x) = e^{-x} + \varepsilon(1-x)e^{-x} + \cdots$$

and in the inner $x = \varepsilon z$,

$$F(z) = 1 + \varepsilon[1 - z - e^{-z}] + \cdots$$

Note: The beauty of the intermediate variable method for matching is that it has so much structure. If you have made any mistakes in solving either inner or outer equation, or if (by chance) you have put the inner region next to the wrong boundary, the structure of the two solutions won't match and you will know something is wrong!

Exercise 10: Look at the problem

$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} = \cos x$$

with boundary conditions $f(0) = 0$, $f(\pi) = 1$. Find the two distinguished stretches for this equation. Calculate the first three terms of the regular expansion, and apply the boundary condition at π to determine the constants.

Now apply your stretch near $x = 0$. Find the first three terms of the inner solution, and apply the boundary condition at $x = 0$ to determine some of the constants in this expansion.

Finally use an intermediate variable to match your two expressions and determine the remaining constants.

5.2 Where is the boundary layer?

In the last example (and in your exercise) we **assumed** the boundary layer would be next to the lower boundary.

If we didn't know, how would we work it out?

Let's start by trying the previous example, but attempting to put the boundary layer near $x = 1$.

Recall we had an outer solution:

$$f(x) \sim a_0 e^{-x} + \varepsilon[a_1 - a_0 x]e^{-x} + \cdots$$

and an inner solution

$$F(z) \sim A_0 + B_0 e^{-z} + \varepsilon[A_1 - A_0 z + B_0(z e^{-z} + e^{-z}) + B_1 e^{-z}] + \cdots$$

with $z = (x - a)/\varepsilon$.

This time we will try to fit the outer solution to the boundary condition at $x = 0$. We have

$$\frac{\mathrm{d}f}{\mathrm{d}x} \sim -a_0 e^{-x} + \varepsilon[a_0 x - a_0 - a_1]e^{-x} + \cdots$$

so the condition is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 0 \qquad \text{at} \qquad x = 0.$$

$$0 = -a_0 + \varepsilon[-a_1 - a_0] + \cdots$$

which gives $a_0 = 0$, $a_1 = 0$ and so on. It is clear that we're not going to get a solution this way!

However, there is another problem, which appears when we try to fit the inner solution at the other boundary. We are setting $a = 1$ and trying to fit $F(z) = e^{-1}$ at $z = 0$. This gives:

$$e^{-1} = A_0 + B_0 + \varepsilon[A_1 + B_0 + B_1] + \cdots$$

so $A_0 = e^{-1} - B_0$ and $A_1 = -B_0 - B_1$. This seems fine, but look at the solution we get:

$$F(z) \sim e^{-1} + B_0(e^{-z} - 1) + \varepsilon[-e^{-1}z + B_0(z - 1 + (z+1)e^{-z}) + B_1(e^{-z} - 1)] + \cdots$$

Remember that, now the boundary layer is at the top, the outer limit of the inner solution will be for large **negative** z: in other words, all of these exponentials will be growing! This can never match onto a well-behaved outer solution.

Key fact: The boundary layer is always positioned so that any exponentials in the inner solution **decay** as you move towards the outer.

5.3 A worse example

This example comes from Cole's book (available in the department library). I would not expect you to solve equations of this difficulty without hints. The governing equation is

$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$

with boundary conditions $f(0) = -1$, $f(1) = 1$.

Note that you have seen this equation before in exercise 9; you should have found the distinguished scaling was $f = \mathrm{ord}(\varepsilon^{1/2})$. Here we can't use that scaling as the boundary conditions fix f to be $\mathrm{ord}(1)$.

Outer

Let us look first at the outer solution. We pose a series

$$f = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \cdots$$

The leading-order equation is

$$f_0 \frac{\mathrm{d}f_0}{\mathrm{d}x} - f_0 = 0 \implies f_0 \left(\frac{\mathrm{d}f_0}{\mathrm{d}x} - 1 \right) = 0$$

which has two solutions,

$$f_0(x) \equiv 0 \qquad \text{and} \qquad f_0(x) = x + C.$$

Note that for both of these, $\mathrm{d}^2 f_0/\mathrm{d}x^2 = 0$ and so f_0 is an exact solution of the equation, and $f_1 = f_2 = \cdots = 0$.

Clearly the branch $f_0 = 0$ can't match either of the boundary conditions, so we know our outer solution must be

$$f(x) = x + C.$$

We have not yet found where the boundary layer will be; since the outer is so simple, we might as well work out the constant for both possibilities now.

If the outer meets $x = 1$ then we have $C = 0$:

$$f_{\text{outer},1}(x) = x.$$

If the outer meets $x = 0$ then instead we have $C = -1$ and

$$f_{\text{outer},0}(x) = x - 1.$$

Inner

What stretch do we expect for the inner? Note that the boundary conditions mean we can't scale f, we can only stretch x. In your solutions to exercise 9, this is equivalent to fixing $\alpha = 0$. There were two balancing scalings that crossed the axis $\alpha = 0$: $\beta = 0$ (which is the outer) and $\beta = 1$, which will be our boundary layer scaling.

[If you didn't do that example, simply find the scaling for $x = a + \varepsilon^b z$ which balances the first two terms of the ODE.]

We introduce $z = (x - a)/\varepsilon$ and rewrite our differential equation:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\frac{\mathrm{d}f}{\mathrm{d}z} - \varepsilon f = 0$$

Now we pose an inner expansion:

$$f \sim F_0(z) + \varepsilon F_1(z) + \varepsilon^2 F_2(z) + \cdots$$

and at leading order the governing equation is

$$\frac{\mathrm{d}^2 F_0}{\mathrm{d}z^2} + F_0\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0.$$

We can integrate this directly once:

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} + \frac{1}{2}F_0^2 = C.$$

Now remember that for a boundary layer solution, we are going to need solutions which decay exponentially to some fixed value out of the layer. This means that as $z \to \pm\infty$ (but not necessarily both), we need $\mathrm{d}F_0/\mathrm{d}z \to 0$ and so $C \geq 0$. Let us set $C = 2k^2$ for convenience.

This ODE for F_0 has three different possible forms of solution. If $k = 0$ the solution is

$$F_0 = \frac{2}{z + C},$$

which is no good as it only decays algebraically. For $k > 0$ there are three solutions, two of which work.

First we look at the possibility that $|F_0| = 2k$. In that case

$$\frac{\mathrm{d}F_0}{\mathrm{d}z} = 0 \qquad F_0 = \pm 2k.$$

This doesn't have the exponential decay we need either. So we move on to the two other cases: $|F_0| < 2k$ and $|F_0| > 2k$.

In the two cases $|F_0| < 2k$ and $|F_0| > 2k$ we can solve the ODE by partial fractions.

If $|F_0| < 2k$ then we have:

$$2\frac{\mathrm{d}F_0}{\mathrm{d}z} = 4k^2 - F_0^2.$$

$$\int \mathrm{d}z = \int \frac{2}{4k^2 - F_0^2}\,\mathrm{d}F_0 = \frac{1}{2k}\int \left(\frac{1}{2k - F_0} + \frac{1}{2k + F_0}\right)\mathrm{d}F_0$$

$$2kz + 2B = -\ln(2k - F_0) + \ln(2k + F_0)$$

$$\exp[2(kz + B)] = \frac{2k + F_0}{2k - F_0} = -1 + \frac{4k}{2k - F_0}$$

$$\frac{4k}{\exp[2(kz + B)] + 1} = 2k - F_0$$

$$F_0 = 2k - \frac{4k}{\exp[2(kz + B)] + 1} = 2k - \frac{4k\exp[-(kz + B)]}{\exp[kz + B] + \exp[-(kz + B)]}$$

$$F_0 = 2k\frac{\exp[kz + B] - \exp[-(kz + B)]}{\exp[kz + B] + \exp[-(kz + B)]} = 2k\frac{\sinh[kz + B]}{\cosh[kz + B]} = 2k\tanh[kz + B].$$

Exercise 11: Show that the solution of the ODE

$$2\frac{\mathrm{d}F_0}{\mathrm{d}z} = 4k^2 - F_0^2$$

with $|F_0| > 2k$ is

$$F_0 = 2k\coth[kz + B].$$

Now we have two solutions which decay exponentially to some limit as $z \to \infty$:

$$F_0 = 2k \tanh [kz + B] \quad \text{and} \quad F_0 = 2k \coth [kz + B].$$

Look at the forms of the tanh and coth curves:

We can see that the tanh solution moves smoothly from one value to another over the width of the boundary layer, whereas the coth profile cannot be given a value $z = 0$. This means that the coth profile can only be used if the boundary layer is at one end or other of the region, whereas the tanh profile can be used anywhere.

Matching

Let us try first to put the boundary layer near $x = 0$. The outer solution must match the boundary condition at $x = 1$ so

$$f_{\text{outer}} = x.$$

Now in the inner region we can either have

$$F(z) = 2k \tanh [kz + B] \quad \text{or} \quad F(z) = 2k \coth [kz + B].$$

In either case we need $F(z = 0) = -1$ and $F(z \to \infty) = 0$. The second of these gives $k = 0$ in both cases, and then we cannot match the other boundary condition for any B. **FAILED.**

Now we try with a boundary layer near $x = 1$. This time the outer solution must match the boundary condition at $x = 0$ so

$$f_{\text{outer}} = x - 1.$$

In the inner region the possibilities are

$$F(Z) = 2k \tanh [kz + B] \quad \text{or} \quad F(z) = 2k \coth [kz + B].$$

The boundary conditions are $F(z = 0) = 1$ and $F(z \to -\infty) = 0$. We have the same problem again: we need both $k \neq 0$ and $k = 0$. **FAILED.**

Finally, let us try having the "boundary layer" in the middle, at some general position a between 0 and 1. This time we have two different branches of the outer solution:

$$f_{\text{outer},1}(x) = x \qquad f_{\text{outer},1}(a) = a.$$

$$f_{\text{outer},0}(x) = x - 1 \qquad f_{\text{outer},0}(a) = a - 1.$$

Our inner solution will then have boundary conditions

$$F(z \to -\infty) = a - 1 \qquad F(z \to \infty) = a.$$

The only profile we are allowed is the tanh profile, which goes from $-2k$ to $2k$ over the width of the layer. This fixes

$$a - 1 = -2k \qquad a = 2k \qquad \implies a = 1/2,\ k = 1/4.$$

Our inner solution is

$$F(z) = \frac{1}{2} \tanh z/4$$

and $z = (x - \frac{1}{2})/\varepsilon$. The complete solution looks like this:

Exercise 12: Consider the **advection-diffusion** equation for f (weak diffusion):

$$\underline{\nabla} \cdot [f\underline{V}] - \varepsilon \nabla^2 f = 0.$$

We will impose boundary conditions

$$\begin{aligned} fV_y - \varepsilon \frac{\partial f}{\partial y} &= 0 \quad \text{at} \quad y = 1, \\ f &= 2 \quad \text{at} \quad y = 2 \end{aligned}$$

The boundary condition at $y = 1$ corresponds to a condition of **no flux** of f through the boundary $y = 1$.

Now suppose that the imposed velocity field is given by

$$V_x = \kappa x / y \qquad V_y = -\kappa.$$

(a) Substitute the velocity field into the governing equation and boundary conditions.

(b) Setting $\varepsilon = 0$, find a solution f_0 which matches the upper boundary condition at $y = 2$. [Hint: try $f_0(x, y) = g(y)$.]

(c) Substitute your solution back into the full governing equation. What can you say about the corrections to f_0 for $\varepsilon \neq 0$?

(d) The inner boundary condition at $y = 1$ is not satisfied. Assume that there is a boundary layer close to $y = 1$. How does the size of this layer scale with ε? Assume that derivatives with respect to x remain order 1.

(e) Introduce a scaled variable z to replace y near $y = 1$. Replace y in the governing equation and give your new PDE to two orders of magnitude. Do the same for the inner boundary condition at $y = 1$.

(f) Using your new PDE and boundary condition **alone**, calculate the first term of a perturbation expansion for $F(x, z) = f(x, y)$, valid within the boundary layer near $y = 1$. You will not be able to determine all the constants (or even all the functions of x) at this stage.

[Hint: if you have a PDE in which all the derivatives are with respect to z, you can solve it like an ODE in z but all the "constants of integration" must be functions of x.]

(g) Can your solution be made to match onto the outer solution as $z \to \infty$ and $y \to 1$?

(h) Continue to the next order correction in your inner expansion. What PDE must be satisfied by the next term? What boundary conditions will be applied to it?

(i) Solve your PDE for the second term in the expansion of the inner solution. Apply the boundary condition to get a relation between the different unknowns.

(j) Carry out matching between your outer and inner solutions. Thus find an ODE in x for the unknown function from the calculation of (f).

(k) Solve the ODE to complete the calculation of the leading-order term in the inner expansion. You may assume that the function is well-behaved at $x = 0$.