7 Solutions

1. The first three terms of the three solutions are:
$$x = 1 + \varepsilon + \frac{3}{2}\varepsilon^2 + \cdots \qquad x = -1 - \frac{1}{3}\varepsilon - \frac{1}{54}\varepsilon^2 + \cdots \qquad x = 2 - \frac{5}{3}\varepsilon - \frac{40}{27}\varepsilon^2 + \cdots$$

2. The first three terms of the solution are
$$y = 1 - \cos x - \sin x + \varepsilon[(x - \pi/2)\sin x + x\cos x] \\ - \varepsilon^2[1 + (x^2/2 - \pi x/2 - 1 + \pi^2/8)\sin x + (x^2/2 - 1)\cos x].$$

3. The root of $\varepsilon x^2 + x - 1 = 0$ near $x = 1$ is $\boxed{x = 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots}$.

4. The distinguished scalings are $\boxed{x \sim \varepsilon,\ x \sim 1, \text{ and } x \sim \varepsilon^{-3},}$ and the three roots are:
$$\boxed{x = -\tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + O(\varepsilon^3) \qquad x = -2 + \tfrac{1}{2}\varepsilon + O(\varepsilon^2) \qquad x = -\varepsilon^{-3} + 2 + O(\varepsilon).}$$

5. The distinguished scalings are $\boxed{x \sim 1 \text{ and } x \sim \varepsilon^{-1},}$ and the three roots are:
$$\boxed{x = -1 - 2\varepsilon + O(\varepsilon^2) \qquad x = -1 \qquad x = -\varepsilon^{-1} + 2 + O(\varepsilon).}$$
The root $x = -1$ has no further terms as it is an exact solution to the full equation.

6. The four roots are: $x = 1 + \frac{1}{3}\varepsilon + \cdots$, $x = -2 - \frac{16}{3}\varepsilon + \cdots$, $x = \varepsilon^{-1/2} + \frac{1}{2} + \cdots$, $x = -\varepsilon^{-1/2} + \frac{1}{2} + \cdots$.

7. The first two terms of the solution are:
$$\boxed{f(x) = A_0 e^x + \varepsilon[B_0 - A_0 x]e^x + \cdots}$$

8. The distinguished stretches are $x = a + \delta X$ with $\boxed{\delta = 1,\ \delta = \varepsilon \text{ and } \delta = \varepsilon^2.}$
The leading order solutions are, respectively,
$$\boxed{f = A_0 e^{-x} \qquad f = B_0 + C_0 e^{-(x-a)/\varepsilon} \qquad f = D_0(x - a)/\varepsilon^2 + E_0 + F_0 e^{-(x-a)/\varepsilon^2}}$$

9. The critical scaling at which all terms balance is $\boxed{\alpha = \beta = 1/2.}$ If we fix $\alpha = 0$ then the two possible balances are between terms I and II, in which case $\beta = -1$, and terms II and III, in which case $\beta = 0$.

10. Distinguished stretches are $\delta = 1$ and $\delta = \varepsilon$.

    Regular expansion:
$$f(x) = 1 + \sin x - \varepsilon[1 + \cos x] - \varepsilon^2 \sin x + \cdots$$
    If $z = x/\varepsilon$ then
$$f(z) = A_0 - A_0 e^{-z} + \varepsilon[A_1 - A_1 e^{-z} + z] + \varepsilon^2[A_2 - A_2 e^{-z}] + \cdots$$
    The matched inner form is
$$f(z) = 1 - e^{-z} + \varepsilon[2e^{-z} - 2 + z] + \cdots .$$

11. Solution is simply a calculation.

12. (a) Governing equation:

$$\kappa \frac{x}{y}\frac{\partial f}{\partial x} - \kappa \frac{\partial f}{\partial y} + \kappa \frac{f}{y} - \varepsilon \nabla^2 f = 0.$$

Boundary conditions:

$$-\kappa f - \varepsilon \frac{\partial f}{\partial y} = 0 \text{ at } y = 1, \qquad f = 2 \text{ at } y = 2$$

(b) $f_0(x, y) = y$.

(c) f_0 is an exact solution to the full equation for any value of ε so there is no need for a correction term.

(d) The boundary layer has size ε.

(e) PDE:

$$\varepsilon^{-1}\left[-\kappa \frac{\partial f}{\partial z} - \frac{\partial^2 f}{\partial z^2}\right] + \kappa x \frac{\partial f}{\partial x} + \kappa f = 0.$$

Inner boundary condition:

$$0 = -\kappa f - \frac{\partial f}{\partial z} \text{ at } z = 0.$$

(f) $F_0(x, z) = B_0(x)e^{-\kappa z}$.

(g) No.

(h) PDE:

$$\frac{\partial^2 F_1}{\partial z^2} + \kappa \frac{\partial F_1}{\partial z} = \left[\kappa x \frac{\mathrm{d}B_0}{\mathrm{d}x} + \kappa B_0(x)\right] e^{-\kappa z}.$$

and the boundary condition:

$$-\kappa F_1 - \frac{\partial F_1}{\partial z} = 0 \text{ at } z = 0.$$

(i)

$$F_1(x, z) = -\left[x \frac{\mathrm{d}B_0}{\mathrm{d}x} + B_0(x)\right] z e^{-\kappa z} + B_1(x)e^{-\kappa z} + A_1(x).$$

Then the boundary condition becomes

$$x \frac{\mathrm{d}B_0}{\mathrm{d}x} + B_0(x) - \kappa A_1(x) = 0.$$

(j) Matching gives

$$A_1(x) = \varepsilon^{-1}.$$

Returning to the boundary condition calculation above, we get

$$x \frac{\mathrm{d}B_0}{\mathrm{d}x} + B_0(x) = \varepsilon^{-1}\kappa.$$

(k) $F(x, z) = \varepsilon^{-1}\kappa e^{-\kappa z}$.