B Similarity solutions

Similarity solutions to PDEs are solutions which depend on certain groupings of the independent variables, rather than on each variable separately. I'll show the method by a couple of examples, one linear, the other nonlinear.

B.1 Linear example: the heat equation

The heat equation in one dimension is

$$u_t = \kappa u_{xx}.$$

This form of equation arises often within boundary layers in a PDE: the first-order derivative may be in an unstretched direction and the higher-order derivative come from the component of ∇^2 in a stretched direction, if the coefficient of ∇^2 in the original equation was small (i.e. an advection–diffusion equation with weak diffusion).

We introduce the dilation transformation

$$z = \varepsilon^a x, \qquad s = \varepsilon^b t, \qquad v = \varepsilon^c u$$

under which $\partial_t = \varepsilon^b \partial_s$ and so on, and the PDE becomes

$$\varepsilon^{b-c} v_s = \kappa \varepsilon^{2a-c} v_{zz}.$$

We look for values under which our PDE is unchanged: in this case we have $b - c = 2a - c$ and so $b = 2a$. That tells us that, provided $b = 2a$, if $u(x,t)$ is a solution of the original equation, then so is $\varepsilon^c u(\varepsilon^a x, \varepsilon^b t)$. But what use is this observation?

The key thing is to note that the combinations

$$vs^{-c/b} = \varepsilon^c u(\varepsilon^b t)^{-c/b} = ut^{-c/b} \quad \text{and} \quad zs^{-a/b} = \varepsilon^a x(\varepsilon^b t)^{-a/b} = xt^{-a/b}$$

are both unchanged by the transformation , which suggests we look for a solution which combines these two forms:

$$u = t^{c/b} f(xt^{-a/b}).$$

Returning to our specific example, we needed $b = 2a$ which means the combination for the argument of f is $xt^{-1/2} = x/\sqrt{t}$. We introduce a new variable for this combination

$$\xi = x/\sqrt{t} \qquad u = t^{c/b} f(\xi)$$

and substitute into the original equation:

$$u_t = \frac{c}{b} t^{c/b-1} f(\xi) + t^{c/b} f'(\xi) \left(-\frac{1}{2} x t^{-3/2}\right) = \left(\frac{c}{b} f(\xi) - \frac{1}{2}\xi f'(\xi)\right) t^{c/b-1}$$

$$u_{xx} = t^{c/b-1} f''(\xi)$$

$$0 = u_t - \kappa u_{xx} = \left(\frac{c}{b} f(\xi) - \frac{1}{2}\xi f'(\xi) - \kappa f''(\xi)\right) t^{c/b-1}$$

We have reduced a constant-coefficient PDE to a variable-coefficient ODE:

$$\kappa f''(\xi) + \frac{1}{2}\xi f'(\xi) - \frac{c}{b} f(\xi) = 0.$$

For a linear equation like this, the ratio c/b is not determined by the equation and we have some flexibility to use in meeting boundary conditions.

B.1.1 Fixed boundary conditions

Suppose our original equation came with the boundary conditions

$$u(x,0) = 0, \quad x > 0 \qquad u(x,t) \to 0, \quad x \to \infty \qquad u(0,t) = u_0, \quad t > 0.$$

Transforming these into the new variables gives

$$t^{c/b} f(\xi) \to 0, \quad \xi \to \infty, \quad \text{even as } t \to 0 \qquad t^{c/b} f(0) = u_0, \quad t > 0.$$

The first of these gives two conditions: $f(\xi) \to 0$ as $\xi \to \infty$ and also $c/b \geq 0$. The second, on the other hand, can only be satisfied if $c/b = 0$ and then we have the transformation

$$u = f(\xi) \qquad \xi = xt^{-1/2}$$

$$\kappa f''(\xi) + \frac{1}{2}\xi f'(\xi) = 0 \qquad f(\xi) \to 0, \quad \text{as } \xi \to \infty, \qquad f(0) = u_0.$$

We can integrate this once to obtain

$$f'(\xi) = C_1 \exp\left[-\frac{\xi^2}{4\kappa}\right]$$

$$f(\xi) = C_1 \int_0^\xi \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2 = C_1(\kappa\pi)^{1/2} \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right) + C_2$$

where $\operatorname{erf}(x) := (2/\sqrt{\pi}) \int_0^x e^{-t^2} \, \mathrm{d}t$. Then the boundary conditions lead to

$$f(\xi) = u_0 \left(1 - \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right)\right) = u_0 \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\kappa}}\right).$$

The solution of the original equation is

$$u = u_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right).$$

B.1.2 Flux boundary conditions

On the other hand, if we have a flux boundary condition on u:

$$u(x,0) = 0, \quad x > 0 \qquad u(x,t) \to 0, \quad x \to \infty \qquad u_x(0,t) = Q, \quad t > 0.$$

then we still have the conditions $c/b \geq 0$ and $f(\xi) \to 0$ as $\xi \to \infty$, but now

$$t^{c/b-1/2} f'(0) = Q, \quad t > 0,$$

which can only be satsfied by taking $c/b = 1/2$. The final transformation is

$$u = t^{1/2} f(\xi) \qquad \xi = x t^{-1/2}$$

giving the ODE and boundary conditions

$$2\kappa f''(\xi) + \xi f'(\xi) - f(\xi) = 0 \qquad f(\xi) \to 0, \quad \xi \to \infty, \qquad f'(0) = Q.$$

It is easy to spot one solution to this equation: $f(\xi) = C_1 \xi$. So we use the reduction-of-order trick and set $f(\xi) = \xi g(\xi)$ to get:

$$2\kappa \xi g''(\xi) + (4\kappa + \xi^2) g'(\xi) = 0$$

Now we can integrate once:

$$g'(\xi) = \frac{C_1}{\xi^2} \exp\left[-\frac{\xi^2}{4\kappa}\right] \qquad g(\xi) = C_1 \int^{\xi} \frac{1}{p^2} \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2$$

$$f(\xi) = C_1 \xi \int^{\xi} \frac{1}{p^2} \exp\left[-\frac{p^2}{4\kappa}\right] \mathrm{d}p + C_2 \xi.$$

Integrating by parts gives

$$f(\xi) = C_1 \left[-\exp\left[-\frac{\xi^2}{4\kappa}\right] - \frac{\xi\sqrt{\pi}}{2\sqrt{\kappa}} \operatorname{erf}\left(\frac{\xi}{2\sqrt{\kappa}}\right)\right] + C_2 \xi,$$

and after applying the boundary conditions the solution becomes

$$f(\xi) = Q\left(\xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\kappa}}\right) - \frac{2\sqrt{\kappa}}{\sqrt{\pi}} \exp\left[-\frac{\xi^2}{4\kappa}\right]\right).$$

B.2 Nonlinear example: KdV equation

The Korteweg–de Vries equation is

$$u_t + 6 u u_x + u_{xxx} = 0.$$

Setting $z = \varepsilon^a x$, $s = \varepsilon^b t$ and $v = \varepsilon^c u$ gives

$$\varepsilon^{b-c} v_s + 6\varepsilon^{a-2c} v v_z + \varepsilon^{3a-c} v_{zzz} = 0,$$

which gives us the conditions for invariance:

$$b - c = a - 2c = 3a - c: \qquad b = 3a, \quad c = -2a.$$

The transformation $u = t^{-2/3} f(\xi)$, $\xi = x t^{-1/3}$ converts the KdV equation to

$$t^{-5/3}\left(f'''(\xi) + f'(\xi)\left[6 f(\xi) - \frac{\xi}{3}\right] - \frac{2}{3} f(\xi)\right) = 0,$$

$$f'''(\xi) + f'(\xi)\left(6 f(\xi) - \frac{\xi}{3}\right) - \frac{2}{3} f(\xi) = 0.$$

This ordinary differential equation can be shown to have the so called **Painlevé property**, meaning that it does not have a movable singular point. A movable singular point is a point where the solution becomes singular, whose location depends on the arbitrary constants of integration. For instance, the equation $y' = y^2$ has the solution $y = (C - \xi)^{-1}$, which has a singular point whose location depends on the arbitrary constant of integration, C. Then this equation does not have the Painlevé property. There is a conjecture[3], that PDEs that reduce to ODEs having the Painlevé property are integrable: that is, they admit soliton solutions and are solvable by the inverse scattering transform.

Thus although we can't solve the ODE above in general, the act of deriving it can give us useful information about the original PDE.

[3] Ablowitz et al., J. Math. Phys. **21**, 715 (1980)