8.5 Diagonalization of symmetric matrices

DEFINITION. *Let A be a square matrix of size n. A is a symmetric matrix if $A^T = A$*

DEFINITION. *A matrix P is said to be orthogonal if its columns are mutually orthogonal.*

DEFINITION. *A matrix P is said to be orthonormal if its columns are unit vectors and P is orthogonal.*

PROPOSITION *An orthonormal matrix P has the property that*

$$P^{-1} = P^T.$$

THEOREM *If A is a real symmetric matrix then there exists an orthonormal matrix P such that*

(i) $P^{-1}AP = D$, where D a diagonal matrix.

(ii) The diagonal entries of D are the eigenvalues of A.

(iii) If $\lambda_i \neq \lambda_j$ then the eigenvectors are orthogonal.

(iv) The column vectors of P are linearly independent eigenvectors of A, that are mutually orthogonal.

EXAMPLE. Recall

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and eigenvectors of A.

(ii) Is A diagonalizable?

(iii) Find an orthonormal matrix P such that $P^TAP = D$, where D is a diagonal matrix.

SOLUTION: We have found the eigenvalues and eigenvectors of this matrix in a previous lecture.

(i), (ii) Observe that A is a real symmetric matrix. By the above theorem, we know that A is diagonalizable. i.e. we will be able to find a sufficient number of linearly independent eigenvectors.

The eigenvalues of A were; $-1, 2$. We found two linearly independent eigenvectors corresponding to $\lambda_1 = -1$: $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

And one eigenvector corresponding to $\lambda_2 = 2$: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(iii) We now want to find an orthonormal diagonalizing matrix P.

Since A is a real symmetric matrix, eigenvectors corresponding to distinct eigenvalues are orthogonal.

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

However the eigenvectors corresponding to eigenvalue $\lambda_1 = -1$, $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are not orthogonal to each other, since we chose them from the eigenspace by making arbitrary choices*. We will have to use Gram Schmidt to make the two vectors orthogonal.

$$\begin{aligned}
\vec{u}_1 &= \vec{v}_1 \\
\text{Proj}_{\vec{u}_1}\vec{v}_2 &= \frac{((-1,1,0).(-1,0,1))}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
\vec{u}_2 &= \vec{v}_2 - \text{Proj}_{\vec{u}_1}\vec{v}_2 \\
&= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}
\end{aligned}$$

We now have a set of orthogonal vectors:

$$\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}\}.$$

We normalize the vectors to get a set of orthonormal vectors:

$$\{\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}\}.$$

We are now finally ready to write the orthonormal diagonalizing matrix:

$$P = \begin{pmatrix} 1\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}$$

and the corresponding diagonal matrix D

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We will now verify that $P^TAP = D$.

$$AP = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 2\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 2\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

$$PD = \begin{pmatrix} 1\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 2\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 2\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$

$$AP = PD \tag{1}$$

Since the columns of P are linearly independent, P has non-zero determinant and is therefore invertible. We multiply (1) by P^{-1} on both sides.

$$\begin{aligned} P^{-1}AP &= P^{-1}PD, \\ &= ID \\ &= D \end{aligned} \tag{2}$$

Also since P is orthonormal, we have

$$P^{-1} = P^T$$

i.e. $PP^T = I = P^TP$.

$$\begin{pmatrix} 1\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1\sqrt{2} & 1/\sqrt{2} & 0 \\ -1\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore from (2) and since $P^{-1} = P^T$ we finally get the relation

$$P^TAP = D.$$

NOTE. *:Look back at how we selected the eigenvecors $\vec{v}_1$ and $\vec{v}_2$; we chose $x_2 = 1, x_3 = 0$ to get $\vec{v}_1$ and $x_2 = 0, x_3 = 1$ to get $\vec{v}_2$. If we had chosen $x_2 = 1, x_3 = 0$ to get $\vec{v}_1$ and $x_2 = -1/2, x_3 = 1$ to get $\vec{v}_2$, then $\vec{v}_1$ and $\vec{v}_2$ would be orthogonal. However it is much easier to make arbitrary choices for x_1 and x_2 and then use the Gram Schmidt Process to orthogonalize the vectors as we have done in this example.