Some irrationality

I realized recently that a few standard irrationality results were taken for granted during the course, but that you may not have seen careful proofs in previous courses. Therefore, I thought I would write them down, so that you can use the results with a clean conscience.

Proposition 1 *Let*

$$r = \pm p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} / q_1^{n_1} q_2^{n_2} \cdots q_t^{n_t} \neq 1$$

be a rational number such that the p_i, q_j are distinct primes and at least one of m_i or n_j is not even. Then $\sqrt{r}$ is irrational.

Proof. By factoring out all possible even prime powers, we can write $r = d^2(a/b)$ where d is rational, $(a, b) = 1$, $a/b \neq 1$, and all the prime factors of a and b occur with multiplicity one. We need only prove that $\sqrt{a/b}$ is irrational. Suppose $\sqrt{a/b} = c/d$ with $(c, d) = 1$. Then $c^2/d^2 = a/b$ and $bc^2 = ad^2$. But then each prime divisor p of c must divide a with multiplicity at least two. Similarly, each prime divisor q of d must divide b with multiplicity at least two. Therefore, the only choice is $c = \pm 1$ and $d = \pm 1$. Then $a/b = (c/d)^2 = 1$.□

As you seem, the key is the unique factorization theorem.

We give some examples of easy corollaries, many variants of which one can easily formulate.

Corollary 2 $\sqrt{3}$ *does not lie in* $\mathbb{Q}(\sqrt{2})$.

Proof. Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$. Then $\sqrt{3} = a\sqrt{2} + b$ with $a, b \in \mathbb{Q}$. So we get $3 = 2a^2 + b^2 + 2ab\sqrt{2}$. Therefore, since $\sqrt{2}$ is irrational, we must have $ab = 0$. If $a = 0$, $\sqrt{3}$ would be rational. If $b = 0$, then $\sqrt{3/2}$ would be rational, yielding a contradiction in any case. □

Corollary 3 $X^2 - 3$ *is irreducible over* $F = \mathbb{Q}(\sqrt{2})$.

The proof is clear from the previous statement.

Corollary 4 $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ *has degree four.*

Proof.

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2$$

□

You can formulate for yourself many other problems of this nature that can be handled using the proposition.