HYPERBOLIC SURFACES AND TEICHMÜLLER SPACES (EXERCISE SHEET)

A reference I like for Teichmuller spaces *A primer on mapping class groups*, freely available at
http://euclid.nmu.edu/ joshthom/Teaching/MA589/farbmarg.pdf

Exercises marked with one star are challenging, those marked with two stars are very difficult.

1. Warm-up exercises

These exercises are a warm-up in Riemann surfaces, hyperbolic geometry and the topology of surfaces.

Exercise 1. *Show that the annuli* $\mathbb{C}/\{z \mapsto z+1\}$ *and* $\mathbb{H}/\{z \mapsto z+1\}$ *are not biholomorphic.*

Exercise 2. *Show that if a group of isometries* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *acts properly discontinuously on* $\mathbb{H}$*, then* $\mathbb{H}/\Gamma$ *is a hyperbolic surface.*

Exercise 3. *Show that the group of biholomorphisms of* $\mathbb{H}$ *is exactly the set of maps of the form*

$$z \longmapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{R}$ *and* $ad - bc > 0$.

Exercise 4. *Let* Σ *be a complete Riemannian surface with constant curvature* -1*. Show that there exists* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *acting properly discontinuously on* $\mathbb{H}$ *such that* Σ *is isometric to* $\mathbb{H}/\Gamma$.

Exercise 5. *Show that the set* $\mathrm{Homeo}_0(\Sigma)$ *of homeomorphisms of a surface* Σ *which are isotopic to the identity is a normal subgroup of* $\mathrm{Homeo}(\Sigma)$

In the rest of the exercise sheet $\mathrm{Mod}(\Sigma)$ denotes the group $\mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma)$. It is often referred to as the **mapping class group**.

Exercise 6. *Assume that* Σ *is the torus. Show that* $\mathrm{Mod}(\Sigma)$ *admits a surjective group homomorphism*

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{SL}(2,\mathbb{Z}).$$

(This homomorphism is actually an isomorphism).

Exercise* 7. *Let* Σ *be a genus* $g \geq 2$ *surface.*

(1) Considering the action of a homeomorphism on $\mathrm{H}_1(\Sigma,\mathbb{Z})$*, show that there is a group morphism*

$$\mathrm{Mod}(\Sigma) \longrightarrow \mathrm{Sp}(2g,\mathbb{Z})$$

where $\mathrm{Sp}(2g,\mathbb{Z})$ *is the group of symplectic matrices with integer coefficients.*

(2) Show that this homomorphism is not injective (consider particular Dehn twists).
(3) (Rather difficult). Show that this homomorphism is surjective (use Dehn twists and generation properties of $\mathrm{Sp}(2g,\mathbb{Z})$*, which you might need to research!).*

Exercise 8. *Find a topological surface* Σ *which carries a Riemann surface structure covered by* $\mathbb{C}$ *and one covered by* $\mathbb{H}$*.*

Exercise 9. *Let* Σ_1 *be a Riemann surface of genus* 1 *and* Σ_g *a Riemann surface of genus* $g \geq 2$*. Show that any holomorphic map*
$f : \Sigma_1 \longrightarrow \Sigma_2$ *is constant.*

2. Chore exercises

2.1. Hyperbolic surfaces.

Exercise 10. *(1) Recall that a triangle* T *in hyperbolic space satisfies that its interior angles sum up to* $\pi - \mathrm{Area}(T)$*. Derive an analogous formula relating the interior angles of an* n*-gon and its area.*
(2) Construct a hyperbolic triangle of interior angles $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$*.*
(3) Using the previous question, construct a hyperbolic octogon all of whose interior angles are $\frac{\pi}{2}$*.*
(4) Show that if (x_1, y_1) *and* (x_2, y_2) *are two pairs of points in* $\mathbb{H}$ *such that* $d(x_1, y_1) = d(x_2, y_2)$*, there exists a unique orientation-preserving isometry* T *such that* $T(x_1) = x_2$ *and* $T(y_1) = y_2$*.*
(5) Derive from the previous questions the construction of a hyperbolic surface of genus 2*.*

Exercise 11. *Show that a subgroup* Γ *of* $\mathrm{PSL}(2,\mathbb{R})$ *acts properly discontinuously on* $\mathbb{H}$ *if and only if it is discrete and doesn't contain elliptic elements other than the identity.*

Exercise 12. *Let* Σ *be a topological surface and let* $\rho : \pi_1\Sigma \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ *be a representation. Show that the two following statements are equivalent*

- ρ *is faithful and* $\rho(\pi_1\Sigma)$ *is a discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*.*
- *The quotient* $\mathbb{H}/\rho(\pi_1\Sigma)$ *is a hyperbolic surface.*

2.2. Genus 1.

Exercise 13 (Genus 1 complex curves)**.** *(1) Let* α *and* β *two complex numbers linearly independent over* $\mathbb{R}$*. Show that the quotient of* $\mathbb{C}$ *by the group* $\mathbb{Z}\cdot\alpha \oplus \mathbb{Z}\cdot\beta$ *acting by translations is a Riemann surface homeomorphic to a 2-torus (the topological space* $S^1 \times S^1$*).*
(2) Let Σ *be a genus* 1 *Riemann surface (that is homeomorphic to a 2-torus). Show that* Σ *is isomorphic to the quotient of* $\mathbb{C}$ *by a group of translation as above.*

(Hint : What are the subgroups of $\mathrm{PSL}(2,\mathbb{R})$ isomorphic to $\mathbb{Z}^2$?)

Exercise* 14 (The modular surface is the moduli space of Riemann surfaces of genus 1)**.** *Consider* $\mathcal{T}_1$ *the Teichmuller space of the genus* 1 *surface.*

(1) Let $\Sigma = S^1 \times S^1$ *be the genus* 1 *compact orientable topological surface and let us endow it with two Riemann surface structures* $\mathcal{R}_1$ *and* $\mathcal{R}_2$*. We have shown in the previous exercise that there exist isomorphisms* φ_1 *(resp.* φ_2*) between* $\mathcal{R}_1$ *(resp.* $\mathcal{R}_2$*) and* $\mathbb{C}/\mathbb{Z}\cdot\alpha_1 \oplus \mathbb{Z}\cdot\beta_1$ *(resp.* $\mathbb{C}/\mathbb{Z}\cdot\alpha_2 \oplus \mathbb{Z}\cdot\beta_2$*). Show that* $\mathcal{R}_1$ *and* $\mathbb{R}_2$ *represent the same point in Teichmüller space if and only if the homology morphisms*

$$(\varphi_1)_* : \pi_1\Sigma \longrightarrow \mathbb{Z}\cdot\alpha_1 \oplus \mathbb{Z}\cdot\beta_1$$

and

$$(\varphi_2)_* : \pi_1\Sigma \longrightarrow \mathbb{Z}\cdot\alpha_2 \oplus \mathbb{Z}\cdot\beta_2$$

if and only if there exists a non-constant complex affine map $f := z \mapsto az+b$ *such that* $(\varphi_2)_* = f \circ (\varphi_1)_*$.

(2) *Show that the Teichmüller space of the genus* 1 *surface is homeomorphic to* $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.
(3) *Let* (α_1, β_1) *and* (α_2, β_2) *two pairs of* $\mathbb{R}$ *linearly-independent, directly oriented vectors in* $\mathbb{C}$. *Show that* $\mathbb{Z}\cdot\alpha_1 \oplus \mathbb{Z}\cdot\beta_1 = \mathbb{Z}\cdot\alpha_2 \oplus \mathbb{Z}\cdot\beta_2$ *if and only if there exists* $A \in \mathrm{SL}(2,\mathbb{Z})$ *such that* $(\alpha_2, \beta_2) = A\cdot(\alpha_1, \beta_1)$.
(4) *Show that the moduli space of genus* 1 *surfaces is homeomorphic to the modular surface*

$$\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$$

where $\mathrm{PSL}(2,\mathbb{Z})$ *acts on* $\mathbb{H}$ *by homographies.*

2.3. **Construction of Teichmuller space and moduli space. The exercises are here to fill the gaps in the construction of $\mathcal{T}_g$ and $\mathcal{M}_g$ discussed in the lecture.**

Exercise* 15 (Pair of pants)**.** *Let* l_1, l_2 *and* l_3 *be three positive real numbers. Construct a hyperbolic metric on a sphere with three boundary components such that:*

(1) *each boundary component is totally geodesic;*
(2) *the respective lengths of the boundary components are* l_1, l_2 *and* l_3.

Such a hyperbolic surface with totally geodesic boundary is called a pair of pants.
Hint : consider a carefully chosen hyperbolic octogon.

Exercise* 16. *Show that the pair of pants with fixed boudary lengths* (l_1, l_2, l_3) *is unique up to isometry.*
Hint : learn a bit of hyperbolic trigonometry.

Exercise 17 (Fenchel-Nielsen coordinates)**.** *Using pairs of pants, find a parametrisation of* $\mathcal{T}_g$ *with* $6g-6$ *parameters for* $g \geq 2$.

Exercise* 18 (Moduli space is an orbifold)**.** *Let* Σ_g *be the compact surface of genus* $g \geq 2$. *We will assume that* $\mathcal{T}_g$ *the Teichmuller space is homeomorphic to* $\mathbb{R}^{6g-6}$.

(1) *Show that* $\mathrm{Mod}(\Sigma_g)$ *acts on* $\mathcal{T}_g$.
(2) *Let* (Σ_g, h) *be a Riemannian metric on* Σ_g *of negative curvature on* Σ_g. *Show that for all* $L > 0$ *the set*

$$\{\gamma \text{ closed geodesic of length less than } L\}$$

is finite.

(3) *Show that the action of* $\mathrm{Mod}(\Sigma_g)$ *on* $\mathcal{T}_g$ *is properly discontinuous.*
(4) *Show that* $\mathcal{M}_g$ *is an orbifold of dimension* $6g-g$.

Exercise* 19. *Show that the moduli space of compact Riemann surfaces of genus* $g \geq 2$ *is not compact by constructing a sequence of Riemann/hyperbolic surfaces of genus* 2 *escaping all compact sets of the moduli space.*

3. Misc

3.1. Algebraic curves.

Exercise 20.** *The goal of this exercise is to show that any Riemann surface S is biholomorphic to a projective curve in $\mathbb{C}P^3$. We assume the existence of a non-constant meromorphic function f.*

1. *Show that f is transcendent on $\mathbb{C}$.*
2. *(That's the difficult part, which you can skip) Show that the degree of any meromorphic function g on $\mathbb{C}(f)$ is finite.* ***(Hint : consider symmetric functions of local inverses of f)***
3. *Show that there is a non-constant holomorphic map $S \longmapsto \mathrm{CP}^2$ mapping S to the zero set of a polynomial.*
4. *(If you know how to blow things up). Show that S is biholomorphic to an a smooth projective curve in $\mathbb{C}P^3$.*

Exercise 21.** *Show that any biholomorphism between smooth projective curves is actually algebraic. (**You can use without proof the fact that any meromorphic function on a smooth projective curve is algebraic, and you can also prove it!**)*

Exercise 22. *Let Σ be a topological surface and let $\rho : \pi_1\Sigma \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ be a representation which is discrete and faithful. Do we necessarily have that $\mathbb{H}/\rho(\pi_1\Sigma)$ is homeomorphic to Σ ?*

Exercise 23. *Let Σ be a **closed** topological surface and let $\rho : \pi_1\Sigma \longrightarrow \mathrm{PSL}(2,\mathbb{R})$ be a representation which is discrete and faithful. Show that $\mathbb{H}/\rho(\pi_1\Sigma)$ is homeomorphic to Σ.*

3.2. Spaces of representations.

Let Γ_g be the fundamental group of a genus $g \geq 2$ compact orientable surface. Recall that Γ_g admits the following presentation

$$\Gamma_g := \langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Exercise* 24. *Character varieties and Teichmüller space*

1. *Show that the set*
$$\{\rho : \Gamma_g \longrightarrow \mathrm{SL}(2,\mathbb{R}) | \rho \text{ is a group homomorphism}\}$$
identifies with an smooth affine variety of $\mathrm{M}(2,\mathbb{R})^g \simeq (\mathbb{R}^4)^g$ of dimension $6g-3$.
2. *Show that the subset χ_{irr} of*
$$\{\rho : \Gamma_g \longrightarrow \mathrm{SL}(2,\mathbb{R}) | \rho \text{ is a group homomorphism}\}$$
made of irreducible representation
3. *Show that the quotient of χ_{irr} by the action of $\mathrm{SL}(2,\mathbb{R})$ by conjugation is a $6g-6$ dimensional manifold.*
4. *Show that $\mathcal{T}_g$ openly injects itself in the quotient of χ_{irr} by the action of $\mathrm{PSL}(2,\mathbb{R})$ by conjugation. (Actually $\mathcal{T}_g$ is one of the finitely many connected components of* $\mathrm{PSL}(2,\mathbb{R})$*, but that's quite hard to prove).*

Exercise* 25. *Let Σ be a sphere with 3 points removed.*

1. *Show that the fundamental group of Σ is the free group on 2 generators.*

(2) *Let a and b be two generators of the fundamental group Σ representing two simple closed curves going around two different punctures. Let ρ_1 and $\rho_2 : \pi_1\Sigma \longrightarrow \mathrm{SL}(2,\mathbb{R})$ such that $\mathrm{Tr}(\rho_1(a)) = \mathrm{Tr}(\rho_2(a))$, $\mathrm{Tr}(\rho_1(b)) = \mathrm{Tr}(\rho_2(b))$ and $\mathrm{Tr}(\rho_1(ab)) = \mathrm{Tr}(\rho_2(ab))$. Show that ρ_1 and ρ_2 are conjugate in $\mathrm{SL}(2,\mathbb{R})$.*

(3) *Show that the space of irreducible representation of $\pi_1\Sigma$ into $\mathrm{PSL}(2,\mathbb{R})$ up to conjugation is naturally isomorphic to an affine variety.* ***Hint: If*** A ***and*** B ***are two matrices in*** $\mathrm{SL}(2,\mathbb{R})$***, there is a polynomial relation between*** $\mathrm{Tr}(A)$***,*** $\mathrm{Tr}(B)$ ***and*** $\mathrm{Tr}(AB)$***.***