Algebraic topology

Spring term 2021-2022

March 25, 2022

Contents

Warning : these notes are a work in progress and some typos and actual mistakes can be lurking round the corner.

If you notice a typo or a mistake in the notes, please kindly send me an email at s.ghazouani@imperial.ac.uk.

- In these notes, all topological spaces X considered will be assumed to be **connected, Hausdorff, path-connected** and **locally path-connected**, unless explicitly mentioned.
- Exercises marked with a star are particularly challenging.

0.1 Reminder about quotient spaces

Let X be a topological space and let $\sim$ be an equivalence relation on X. Let

$$X/_\sim$$

be the set of equivalence classes for $\sim$. Denote by

$$\begin{array}{rcccc} \pi & := & X & \longrightarrow & X/_\sim \\ & & x & \longmapsto & [x] \end{array}$$

the natural projection.

We define a topology on $X/_\sim$ the following way:

$$U \subset X/\sim \text{ is open if } \pi^{-1}(U) \text{ is open.}$$

Proposition 1. *The set of such Us defines a topology on $X/_\sim$.*

Proof. We've got three things to check.

1. $\emptyset$ is open as it is $\pi^{-1}(\emptyset)$ and so is $X/_\sim$ as $X = \pi^{-1}(X/_\sim)$ is open.
2. Let U_1 and $U_2 \subset X/_\sim$ two open sets. Since $\pi^{-1}(U_1 \cap U_2) = \pi^{-1}(U_1) \cap \pi^{-1}(U_2)$, $\pi^{-1}(U_1 \cap U_2)$ is open therefore $U_1 \cap U_2$ is open.
3. Let $(U_i)_{i \in I}$ an arbitrary family of open sets of $X/_\sim$. Since $\pi^{-1}(\bigcup_i U_i) = \bigcup \pi^{-1}(U_i)$, $\pi^{-1}(\bigcup_i U_i)$ is open, and thus so is $\pi^{-1}(\bigcup_i U_i)$.

This completes the proof of the proposition. □

Note that in particular with this topology the projection π is continuous.

The main property of quotient spaces that we are going to make extensive use of is the following.

Proposition 2. *Let $X/_\sim$ be a quotient space and Y a topological space. A map $f : X/_\sim \longrightarrow Y$ is continuous if and only if $f \circ \pi : X \longrightarrow Y$ is continuous.*

Proof. If f is continuous, so is $f \circ \pi$ as a composition of continuous maps.

Assume that $f \circ \pi$ is continuous. Let U be an open set in Y. $\pi^{-1}(f^{-1}(Y)) = (f \circ \pi)^{-1}(Y)$ and since $f \circ \pi$ is continuous, $\pi^{-1}(f^{-1}(Y))$ is open. Consequently, $f^{-1}(Y)$ is open. This implies that f is continuous. □

Exercise 1. *Let $\sim$ be the equivalence relation on $[0,1]$ where $0 \sim 1$ and if $x \neq 0, 1$, x is alone in its equivalent class. Show that $[0,1]/_{\sim}$ is homeomorphic to $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.*

To solve Exercise 1, one must use the following Proposition.

Proposition 3. *Let X be a compact space and Y Hausdorff. Any map*

$$f : X \longrightarrow Y$$

that is continuous and bijective is a homeomorphism.

Exercise 2. *Give a proof of Proposition 3.*

0.2 Basic notions and examples

(This paragraph covers the material discussed in Lecture 2.)

0.2.1 Topological manifolds

In this paragraph we introduce the notion of (topological) *manifolds* that will not be central to this course. It can be skipped by a reader only willing to learn the technical aspects of algebraic topology. However, the study of manifolds has been one of the main motivations for the introduction of algebraic topology in the early 20^{th} century and it felt appropriate to start with a quick definition and a few examples.

Definition 1. *A topological manifold of dimension n is a Hausdorff space X such that*

- *(X is locally Euclidean) for all $x \in X$, there exists a neighbourhood $U_x \subset X$ and a homeomorphism $\varphi_{U_x} : U_x \longrightarrow \mathbb{D}^n$;*
- *(second-countable) X has a countable and dense subset.*

1. $\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ is a manifold.
2. One can check that $S^1 := \{x \in \mathbb{R}^2 \mid ||x|| = 1\}$ is a 1-dimensional manifold.
3. The sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ is a n-dimensional manifold.
4. The two-dimensional torus, which is the set of points in $\mathbb{R}^3$

$$\{(\cos\theta(2+\cos\varphi), \sin\theta(2+\cos\varphi), \sin\varphi \mid (\theta,\varphi) \in \mathbb{R}^2\}.$$

5. The compact surface of genus $g > 0$.

Exercise 3. *Show that S^1 is 1-dimensional manifold.*

Hint: *Use the inverse of the map $\theta \mapsto (\cos\theta, \sin\theta)$ on open sets where it is defined.*

Exercise* 4. *Show that the sphere S^2 is a 2-dimensional manifold.*

0.2.2 Path and loops, simple-connectedness

Definition 2 (Loop). *Let X be a topological space. A **loop** (or **closed path**) in X is a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = \gamma(1)$.*

- We say that two loops γ_0 and γ_1 are **freely homotopic** if there exists a continuous

$$\begin{array}{cccc} \gamma & : & [0,1]\times[0,1] & \longrightarrow & X \\ & & (t,u) & \longrightarrow & \gamma(t,u) \end{array}$$

  such that for all $t \in [0,1]$, $\gamma(t,0) = \gamma(t,u)$ (in other words for all t, $\gamma(t,\cdot)$ is a loop) and such that $\gamma(0,\cdot) = \gamma_0$ and $\gamma(1,\cdot) = \gamma_1$.

- A loop γ is said to be **homotopically trivial** if it is freely homotopic to a constant loop (*i.e.* a loop γ such that for all $u \in [0,1]$, $\gamma(u) = x$ for some $x \in X$).

Definition 3 (Simple connectedness). *A topological space X is said to be **simply connected** if any loop in X is homotopically trivial.*

Exercise 5. *Show that $\mathbb{R}^n$ is simply connected for all n.*

Exercise 6. *Show that the set*

$$\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$$

is simply connected.

Exercise 7. *Show that a topological space X is simply-connected if and only if any continuous map $f : S^1 \longrightarrow X$ extends to a continuous $\bar{f} : \bar{\mathbb{D}}^2 \longrightarrow X$.*

Exercise* 8. *Show that the sphere S^2 is simply connected.*

(This important exercise will be treated in a live session)

Historical note about simply-connected spaces Simply-connected spaces play an important role as we will see that any "reasonable" topological space X is covered by a simply connected space (its *universal cover* $\tilde{X}$) and that the information missing to recover X from $\tilde{X}$ is a group $\pi_1(X)$, the *fundamental group* of X. The problem of classifying simply-connected space (topological manifolds in particular) has been driving research in topology from the early 20^{th} century onwards. We give two famous (and difficult) theorem that illustrate this theme.

Theorem 4. *S^2 and $\mathbb{R}^2$ are, up to homeomorphism, the only two simply-connected 2-dimensional manifolds.*

Theorem 5 (Poincaré conjecture). *The only compact, simply-connected 3-dimensional manifold is the sphere S^3.*

This last theorem is the famous *Poincaré conjecture*, which was suggested as an open problem by Henri Poincaré in 1904 and finally proved by Grigori Perelman in 2003.

Exercise 9. *Show that there are infintely many (non-compact) simply-connected 3-dimensional manifolds.*

0.2.3 First examples

The circle The circle S^1 is defined as the set of points in $\mathbb{C}$ of modulus 1. It is a compact, one-dimensional manifold.

Spheres The n-dimensional sphere S^n is defined as the set of points in $\mathbb{R}^n$ of Euclidean norm 1 *i.e.*

$$S^n := \left\{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\right\}.$$

Tori The n-dimensional torus is the topological space

$$\mathbb{T}^n := \left(S^1\right)^n.$$

Wedge of circles Consider two copies of S^1, which we denote by S_a^1 and S_b^1. Pick two arbitrary points $x_a \in S_a^1$ and $x_b \in S_b^1$. Define the equivalence relation $\sim$ of $S_a^1 \cup S_b^1$ by $x_a \sim x_b$ and such that any $x \neq x_a, x_b$ is alone in its equivalence class.

By definition

$$S^1 \vee S^1 = S_a^1 \cup S_b^1 /_{\sim}.$$

Intuitively $S^1 \vee S^1$ is just a pair of circles glued together at a single point.

Projective spaces Recall that $S^n := \left\{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\right\}$. Let $\sim$ be the equivalence relation on S^n defined by $x \sim -x$. We define

$$\mathbb{RP}^n := S^n /_{\sim}.$$

Exercise 10. *Consider the equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ defined by*

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{ such that } y = \lambda x.$$

Show that $(\mathbb{R}^{n+1} \setminus \{0\})/_{\sim}$ is homeomorphic to $\mathbb{RP}^n$.

Chapter 1

Fundamental group and covering spaces

1.1 The 2-dimensional torus

This paragraph covers the material discussed in Lectures 3 and 4

In this section we describe in detail the case of the two dimensional torus. We show it can be described as the quotient of a natural group action on $\mathbb{R}^2$.

1.1.1 $\mathbb{T}^2$ as a quotient space

We define Γ to be the subgroup of $\mathrm{Homeo}(\mathbb{R}^2)$ generated by the two following elements

$$t_1 := (x_1, x_2) \longmapsto (x_1 + 1, x_2)$$

and

$$t_2 := (x_1, x_2) \longmapsto (x_1, x_2 + 1).$$

Proposition 6. *The group Γ is isomorphic to $\mathbb{Z}^2$.*

Proof. First note that t_1 and t_2 commute, *i.e.* $t_1 \circ t_2 = t_2 \circ t_1$. It thus follows that the map

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ (n,m) & \longmapsto & t_1^n \circ t_2^m \end{array}$$

induces a group homomorphism. Since t_1 and t_2 commute, any element of Γ can be written as $t_1^n \circ t_2^m$ for some pair $(n,m) \in \mathbb{Z}^2$. The above map is thus onto. Finally, one easily check that $t_1^n \circ t_2^m = (x_1, x_2) \mapsto (x_1 + n, x_2 + m)$, the kernel of the map above is then trivial. The map

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \\ (n,m) & \longmapsto & t_1^n \circ t_2^m \end{array}$$

is a group homomorphism, which terminates the proof of the Proposition. $\square$

We now define the following equivalence relation: $(x,y) \sim (u,v)$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma(x,y) = (u,v)$. In other words, $(x,y) \sim (u,v)$ if and only $\exists (n,m) \in \mathbb{Z}^2$ such that $(u,v) = (x,y) + (n,m)$. We let the reader check that $\sim$ is an equivalence relation. We set the following notation

$$\mathbb{R}^2/\Gamma = \mathbb{R}^2/\mathbb{Z}^2 := \mathbb{R}^2/_{\sim}.$$

Proposition 7. *The quotient space* $\mathbb{R}^2/\mathbb{Z}^2 := \mathbb{R}^2/\Gamma$ *is homeomorphic to* $\mathbb{T}^2 := S^1 \times S^1$.

Proof. Consider the formula $\psi(x_1, x_2) = (e^{2\pi i x_1}, e^{2\pi i x_2})$ which defines a continuous map $\psi : \mathbb{R}^2 \longrightarrow S^1 \times S^1$. One easily checks, using elementary properties of the exponential map, that we have the following

$$\psi(x_1, x_2) = \psi(y_1, y_2) \text{ if and only if there exists } t \in \Gamma \text{ such that } (y_1, y_2) = t(x_1, x_2).$$

It ensues that ψ induces an injective map

$$\phi : \mathbb{R}^2/\Gamma \longrightarrow S^1 \times S^1$$

that is such that $\psi = \phi \circ \pi$. Since ψ is surjective, so is ϕ.

Consider the square $S = [0,1] \times [0,1] \subset \mathbb{R}^2$. Since any element of $\mathbb{R}^2$ can be written as $(n,m) + (x,y)$ where $n,m) \in \mathbb{Z}^2$ and $(x,y) \in S$, the projection $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\Gamma$ restricted to S is surjective. Since S is compact, it ensues that $\pi(S) = \mathbb{R}^2/\Gamma$ is compact.

Now $\phi : \mathbb{R}^2/\Gamma \longrightarrow S^1 \times S^1$ is a bijective continuous map between compact spaces, it follows that ϕ is a homeomorphism.

□

We now consider the projection

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}^2.$$

We show that it has the following important property.

Proposition 8. *For any* $x \in \mathbb{R}^2$, *the restriction of* π *to* $B(x, \frac{1}{4})$ *is a homeomorphism onto its image.*

Proof. If x and y in $\mathbb{R}^2$ are such that $\pi(x) = \pi(y)$, we must have $x - y \in \mathbb{Z}^2$. So unless $x = y$ we have $d(x,y) \geq 1$. $\forall y, z \in \overline{B(0, \frac{1}{4})}, d(x,y) \leq \frac{1}{2}$ which implies that π is injective restricted to $\overline{B(0, \frac{1}{4})}$. Since $\overline{B(0, \frac{1}{4})}$ is compact, π restricted to $\overline{B(0, \frac{1}{4})}$ is homeomorphism onto its image. It is thus the case of π restricted to any subset of $\overline{B(0, \frac{1}{4})}$, which terminates the proof.

□

1.1.2 Lifts of paths and applications

Proposition 9 (Lifts of paths). *Let $\gamma : [0,1] \longrightarrow \mathbb{T}^2$ be a continuous path such that $\gamma(0) = x$. For any $\tilde{x} \in \mathbb{R}^2$ such that $\pi(\tilde{x}) = x$, there exists a unique path $\widetilde{\gamma} : [0,1] \longrightarrow \mathbb{R}^2$ such that*

- $\widetilde{\gamma}(0) = \tilde{x}$;
- $\pi \circ \widetilde{\gamma} = \gamma$.

Proof. We start by showing the **existence** of such a $\tilde{\gamma}$. By compactness of $[0,1]$ and continuity of γ, there exist $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ such that

$$\forall k \leq n, \gamma([t_k, t_{k+1}]) \subset B(x_k, \frac{1}{4})$$

where $x_k = \gamma(t_k)$.

• We define $\tilde{\gamma}$ the following way : recall that $\gamma(0) = x$. Since $\gamma([0,t_1]) \subset B(\gamma(0), \frac{1}{4})$ we can consider $\psi^{-1} \circ \gamma : [0,t_1] \longrightarrow \mathbb{R}^2$ where ψ is the restriction of π to $B(\tilde{x}, \frac{1}{4})$, which is a homeomorphism thanks to the previous Proposition. By definition, $\psi^{-1} \circ \gamma$ lifts $\gamma_{|[0,t_1]}$ to $\mathbb{R}^2$.

• Now assume that we have successfully constructed $\tilde{\gamma_k} : [0,t_k] \longrightarrow \mathbb{R}^2$ such that $\pi \circ \tilde{\gamma_k} = \gamma_{|[0,t_k]}$ and $\tilde{\gamma_k}(0) = \tilde{x}$. Let $\tilde{x_k} := \tilde{\gamma_k}(t_k)$ and let ψ_k be the restriction of π to $B(\tilde{x_k}, \frac{1}{4})$. Since $\gamma([t_k, t_{k+1}]) \subset B(x_k, \frac{1}{4}) = \psi_k(B(\tilde{x_k}, \frac{1}{4}))$ and ψ_k is a homeomorphism onto its image, we can define

$$\tilde{\gamma_{k+1}} := \begin{array}{c} \tilde{\gamma_k}(t) \text{ if } t \in [0, t_k] \\ \psi_k^{-1}(\gamma(t)) \text{ if } t \in (t_k, t_{k+1}] \end{array}.$$

Since $\psi_k^{-1}(\gamma(t))$ varies continuously with t and $\psi_k^{-1}(\gamma(t))$ tends to $\tilde{x_k} = \tilde{\gamma_k}(t_k)$ when t tends to t_k, $\tilde{\gamma_{k+1}}$ is continuous. The process can thus be extended until $k = n$ which proves the existence of $\tilde{\gamma}$.

We now deal with the **uniqueness**. Consider $\tilde{\gamma}$ and $\tilde{\gamma}'$ two such lifts and let

$$\delta := \tilde{\gamma} - \tilde{\gamma}'.$$

Since $\tilde{\gamma}(0) = \tilde{\gamma}'(0) = \tilde{x}$, $\delta(0) = 0$. Moreover, since $\pi \circ \tilde{\gamma} = \pi \circ \tilde{\gamma}$, for all t we have $\delta(t) = \tilde{\gamma}(t) - \tilde{\gamma}'(t) \in \mathbb{Z}^2$. δ is thus a continuous function taking values in a discrete set. $[0,1]$ being connected, δ must be constant and thus equal to 0 which implies $\tilde{\gamma} = \tilde{\gamma}'$.

□

We now turn to the question of classifying closed paths on $\mathbb{T}^2$ up to homotopy. Let γ be a closed path, consider any lift $\tilde{\gamma} : [0,1] \longrightarrow \mathbb{R}^2$. Since γ is closed, $\gamma(0) = \pi \circ \tilde{\gamma}(0) = \gamma(1) = \pi \circ \tilde{\gamma}(1)$ therefore $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^2$. Since any two lifts of γ differ by a constant in $\mathbb{Z}^2$, the quantity $\tilde{\gamma}(1) - \tilde{\gamma}(0)$ does not depend upon the choice of the lift $\tilde{\gamma}$ but just on γ itself. We define thus

$$\rho(\gamma) := \tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^2.$$

Proposition 10. *If γ_0 and γ_1 are freely homotopic, we have*

$$\rho(\gamma_0) = \rho(\gamma_1).$$

Proof. Let $\gamma : [0,1] : \times[0,1] \longrightarrow \mathbb{T}^2$ a free-homotopy between γ_0 and γ_1. We use the notation $\gamma_u(t) = \gamma(u,t)$, this way $(\gamma_u)_{u\in[0,1]}$ is a family of paths interpolating between γ_0 and γ_1.

Let $\delta : [0,1] \longrightarrow \mathbb{T}^2$ be the path defined by $\delta(u) = \gamma_u(0)$ (that is the path describing the base points of the γ_us). Consider an arbitrary lift of $\tilde{\delta} : [0,1] \longrightarrow \mathbb{R}^2$. Now for any u, consider the lift $\tilde{\gamma_u}$ of γ_u such that $\tilde{\gamma_u}(0) = \delta_u$. The map

$$u \longmapsto \tilde{\gamma_u}((1) - \tilde{\gamma_u}(0)$$

is continuous[1] and since it take value in $\mathbb{Z}^2$ it is constant. Thus

$$\rho(\gamma_0) = \tilde{\gamma_0}((1) - \tilde{\gamma_0}(0) = \tilde{\gamma_1}((1) - \tilde{\gamma_1}(0) = \rho(\gamma_1).$$

□

Let $\mathcal{P}$ be the set of loops in $\mathbb{T}^2$. We endow it with the equivalence relation

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1 \text{ and } \gamma_2 \text{ are freely homotopic.}$$

We set

$$\mathcal{L} = \mathcal{P}/_{\sim}$$

that is $\mathcal{L}$ is the set of free-homotpy classes of closed paths on $\mathbb{T}^2$. We concluded our study of $\mathbb{T}^2$ by a description of $\mathcal{L}$.

Proposition 11. *Free-homotopy classes of curves in $\mathbb{T}^2$ are in one-to-one correspondence with elements of $\mathbb{Z}^2$.*

Proof. The map $\gamma \longmapsto \rho(\gamma)$ is constant on free-homotopy classes (as indicates the previous proposition) and therefore defines

$$\rho : \mathcal{L} \longrightarrow \mathbb{Z}^2.$$

We want to show that this map is actually a bijection.

Injectivity. Let γ_0 and γ_1 two closed paths such that $\rho(\gamma_0) = \rho(\gamma_1)$. This means that if one takes any two lifts $\tilde{\gamma_0}, \tilde{\gamma_1} = [0,1] \longrightarrow \mathbb{R}^2$, we have

$$\tilde{\gamma_0}(1) - \tilde{\gamma_0}(0) = \tilde{\gamma_1}(1) - \tilde{\gamma_1}(0) \in \mathbb{Z}^2.$$

We thus consider

$$\tilde{\gamma_t} = (1-t)\gamma_0 + t\gamma_1.$$

[1] Although quite intuitive, this statement would deserve a rigorous justification, which we will do in the next chapter about covering maps.

One easily checks that $\tilde{\gamma}_t(1) - \tilde{\gamma}_t(0) = \tilde{\gamma}_0(1) - \tilde{\gamma}_0(0)$ for all $t \in [0,1]$ thus $\pi \circ \tilde{\gamma}_t : [0,1] \longrightarrow \mathbb{T}^2$ is a closed path for all t. This proves that γ_0 and γ_1 are freely-homotopic. $\rho : \mathcal{L} \longrightarrow \mathbb{Z}^2$ is therefore injective.

Surjectivity. For any $(n, m) \in \mathbb{Z}^2$ the path $\gamma := t \mapsto (tn, tm)$, $t \in [0,1]$ projects to a closed path in $\mathbb{T}^2$ such that $\rho(\gamma) = (n, m)$.

□

1.1.3 Summary

In this paragraph we have done the following with the torus $\mathbb{T}^2$.

1. We have identified a **simply-connected** space ($\mathbb{R}^2$) sitting on top of it (that is there is a projection from $\mathbb{R}^2$ to $\mathbb{T}^2$ that is a local homeomorphism).
2. The lack of injectivity of this projection is encoded in a **discete group** ($\mathbb{Z}^2$), and $\mathbb{T}^2$ can be realised as a quotient of $\mathbb{R}^2$ by an action of $\mathbb{Z}^2$.
3. This discrete group $\mathbb{Z}^2$ can in some way be but in correspondence with **loops** in $\mathbb{T}^2$ (up to continuous deformation).

We will see in the next sections that this description generalises to a large class of topological spaces, where the simply-connected space sitting on top of X will be called the *universal cover* and the discrete group acting and representing closed paths in X will be called its *fundamental group*.

Exercise 11. *Find an example of a continuous bijection between two topological spaces that is NOT a homeomorphism.*

1.2 The fundamental group

This paragraph covers the material discussed in Lecture 5.

1.2.1 Homotopy between paths

Let X be a topological space and $x_0 \in X$.

Definition 4 (Homotopy with fixed end points)**.** *Two paths $\gamma_0, \gamma_1 : [0,1] \longrightarrow X$ going from x to y (that is $\gamma_0(0) = \gamma_1(0) = x$ and $\gamma_0(1) = \gamma_1(1) = y$) are said to be* ***homotopic with fixed end points*** *if there exists a continuous*

$$\begin{array}{rccc} H & : & [0,1] \times [0,1] & \longrightarrow & X \\ & & (t,u) & \longrightarrow & \gamma(t,u) \end{array}$$

such that for all $t \in [0,1]$, $H(t,0) = x$, $H(t,1) = y$ (in other words for all t, $H(t,\cdot)$ goes from x to y) and such that $H(0,\cdot) = \gamma_0$ and $H(1,\cdot) = \gamma_1$.

Proposition 12. *The relation "be homotopic with fixed end points" is an equivalence relation on the set of paths in X.*

Proof. We write $\gamma_1 \sim \gamma_2$ if γ_1 is homotopic with fixed endpoints to γ_2.

- Reflexivity. For a path γ, γ is homotopic with fixed to γ via $H(t,x) = \gamma(x)$ for all t.
- Symmetry. $\gamma_1 \sim \gamma_2$ and H is a homotopy such that $H(0,\cdot) = \gamma_1$ and $H(1,\cdot) = \gamma_2$, the homotopy $H'(t,x) = H(1-t,x)$ shows that $\gamma_2 \sim \gamma_1$.
- Transitivity. If we have got $\gamma_1 \sim \gamma_2$ via H and $\gamma_2 \sim \gamma_3$ via H', we define

$$\begin{array}{rcccl} J & := & [0,1]\times[0,1] & \longrightarrow & X \\ & & (t,u) & \longmapsto & H(2t,u) \quad \text{if } t \leq \frac{1}{2} \\ & & (t,u) & \longmapsto & H'(2t-1,u) \quad \text{if } t \geq \frac{1}{2} \end{array}.$$

  J is a homotopy fixed fixed endpoints between γ_1 and γ_3.

□

Exercise 12. *Show that a path-connected space X is simply-connected if and only if for any two points $x, y \in X$ any two paths whose end points are x and y are homotopic with fixed end points.*

Concatenation Let $\gamma : [0,1] \longrightarrow X$ be a path from x to y and $\nu : [0,1] \longrightarrow X$ a path from y to z. We define

$$\begin{array}{rcccc} \nu * \gamma & := & [0,1] & \longrightarrow & X \\ & & t \in [0,\frac{1}{2}] & \longmapsto & \gamma(2t) \\ & & t \in [\frac{1}{2},1] & \longmapsto & \nu(2t-1) \end{array}.$$

One easily checks that $\nu * \gamma$ is a path form x to z.

Proposition 13. *Let γ_1 and γ_2 be two homotopic paths from x to y and ν_1 and ν_2 two homotopic paths from y to z. Then the paths $\nu_1 * \gamma_1$ and $\nu_2 * \gamma_2$ are homotopic.*

Proof. Let H be a homotopy between γ_1 and γ_2 and J be a homotopy between ν_1 and ν_2. We introduce

$$\begin{array}{rcccl} K & := & [0,1]\times[0,1] & \longrightarrow & X \\ & & (t,u) & \longmapsto & H(t,2u) \quad \text{if } u \leq \frac{1}{2} \\ & & (t,u) & \longmapsto & J(t,2u-1) \quad \text{if } u \geq \frac{1}{2} \end{array}.$$

Note that $K(0,\cdot) = \nu_1 * \gamma_1$ and $K(1,\cdot) = \nu_2 * \gamma_2$. Since for all t, $H(t,1) = J(t,0) = y$, K is continuous and is a homotopy with fixed endpoints between $\nu_1 * \gamma_1$ and $\nu_2 * \gamma_2$.

□

1.2.2 The fundamental group

Let X be a topological space that is **path-connected** and consider $x_0 \in X$. We define

$$\pi_1(X, x_0) = \{\text{loops } \gamma \text{ based at } x_0\}/_\sim.$$

In other words, $\pi_1(X, x_0)$ is the set of equivalence classes of closed paths based at x_0 for the equivalence relation "be homotopic with fixed end points".

Group law By Proposition 13, if γ_1 and γ_2 are two loops based at x_0, the class $[\gamma_1 * \gamma_2]$ in $\pi_1(X, x_0)$ only depends on the classes of γ_1 and γ_2. We can therefore define an operation $*$ on $\pi_1(X, x_0)$ by saying that $[\gamma_1] * [\gamma_2] = [\gamma_1 * \gamma_2]$ for any arbitrary representatives γ_1 and γ_2 of $[\gamma_1]$ and $[\gamma_2]$ respectively.

Proposition 14. *$\pi_1(X, x_0)$ endowed with $*$ is a group.*

Proof. We have three things to check.

The law $*$ is associative. This is the most delicate bit. We want to show that for paths γ_1, γ_2 and γ_3 based at x_0, $(\gamma_1 * \gamma_2) * \gamma_3$ is homotopic to $\gamma_1 * (\gamma_2 * \gamma_3)$. We achieve this point by showing that these two paths are equal up to reparametrisation. By definition

$$\begin{array}{rcll}(\gamma_1 * \gamma_2) * \gamma_3(t) & = & \gamma_1(4t) & \text{for } t \in [0, \frac{1}{4}] \\ & = & \gamma_2(4t-1) & \text{for } t \in [\frac{1}{4}, \frac{1}{2}] \\ & = & \gamma_3(2t-1) & \text{for } t \in [\frac{1}{2}, 1]\end{array}$$

and

$$\begin{array}{rcll}\gamma_1 * (\gamma_2 * \gamma_3)(t) & = & \gamma_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ & = & \gamma_2(4t-2) & \text{for } t \in [\frac{1}{2}, \frac{3}{4}] \\ & = & \gamma_3(4t-3) & \text{for } t \in [\frac{3}{4}, 1]\end{array}$$

One can check that if one sets

$$\begin{array}{rcll}\varphi(t) & = & 2t & \text{for } t \in [0, \frac{1}{4}] \\ & = & t + \frac{1}{4} & \text{for } t \in [\frac{1}{4}, \frac{1}{2}] \\ & = & \frac{1}{2}t + \frac{1}{2} & \text{for } t \in [\frac{1}{2}, 1]\end{array}$$

we have $\gamma_1 * (\gamma_2 * \gamma_3)(t) = (\gamma_1 * \gamma_2) * \gamma_3 \circ \varphi(t)$. Therefore the homotopy defined by the formula

$$H(s, t) = (\gamma_1 * \gamma_2) * \gamma_3(s\varphi(t) + (1-s)t)$$

shows that $(\gamma_1 * \gamma_2) * \gamma_3$ and $\gamma_1 * (\gamma_2 * \gamma_3)$ are homotopic.

Neutral element. We claim that the class of the path $e := t \mapsto x$ is the neutral class. To see this it suffices to check that for any γ (based at x_0), $\gamma * e$ (and $e * \gamma$) are homotopic to γ. We just check it for $\gamma * e$ as the case of $e * \gamma$ is almost identical. Let $\varphi_s(t) = (s+1)t$ for $t \leq \frac{1}{s+1}$ and $\varphi_s(t) = 1$ for $t \geq \frac{1}{s+1}$.

$$H(s, t) := \gamma \circ \varphi_s(t)$$

defines a homotopy between γ and $\gamma * e$.

Existence of the inverse. Let γ be a path. We check that the inverse of $[\gamma]$ is the class of the path $\delta := t \mapsto \gamma(1-t)$. It suffices to check that $\delta * \gamma$ is homotopic to e. We define in a similar fashion to the previous points

$$\begin{array}{rcll} \varphi_s(t) & = & 2t & \text{for } t \in [0, \frac{s}{2}] \\ & = & s - 2t & \text{for } t \in [\frac{s}{2}, s] \\ & = & 0 & \text{for } t \in [s, 1] \end{array}$$

We obtain that

$$H(s,t) := \gamma\varphi_s(t)$$

realises a homotopy between $\gamma * \delta$ and e which proves that $[\delta]$ is the inverse of $[\gamma]$.

□

Exercise 13. *Show that a path-connected space X is simply connected if and only if $\pi_1(X, x_0)$ is trivial for any x_0.*

Dependence on the base point To define $\pi_1(X, x_0)$ we have had to choose a base point. The next proposition shows that to some extent $\pi_1(X, x_0)$ does not really depend on this choice.

Proposition 15. *Let x_0 and x_1 be two points in X and let δ be a path from x_0 to x_1. Then the map*

$$\begin{array}{ccc} \pi_1(X, x_0) & \longrightarrow & \pi_1(X, x_1) \\ [\gamma] & \longmapsto & [\delta * \gamma * \delta^{-1}] \end{array}$$

is a group homomorphism, where $\delta^{-1} := t \mapsto \delta(1-t)$.

Proof. The map $[\gamma] \longmapsto [\delta * \gamma * \delta^{-1}]$ send the neutral element $[e]$ on $[\delta * e * \delta^{-1}]$ which is equal to $[\delta] * [e] * [\delta^{-1}] = [\delta * \delta^{-1}]$. Using the same homotopy as in the proof of the previous Proposition, we have that $\delta * \delta^{-1}]$ is homotopic to the constant path based at x_1.

Furthermore, for any γ_1 and γ_2 based at x_0 we have

$$[\delta * \gamma_1 * \delta^{-1}] * [\delta * \gamma_2 * \delta^{-1}] = [\delta * \gamma_1 * \delta^{-1} * \delta * \gamma_2 * \delta^{-1}]$$

$$[\delta * \gamma_1 * \delta^{-1}] * [\delta * \gamma_2 * \delta^{-1}] = [\delta * \gamma_1] * [\delta^{-1} * \delta] * [\gamma_2 * \delta^{-1}]$$

since $[\delta^{-1} * \delta] = [e]$ we get

$$[\delta * \gamma_1 * \delta^{-1}] * [\delta * \gamma_2 * \delta^{-1}] = [\delta * \gamma_1] * [\gamma_2 * \delta^{-1}]$$

$$[\delta * \gamma_1 * \delta^{-1}] * [\delta * \gamma_2 * \delta^{-1}] = [\delta * (\gamma_1 * \gamma_2) * \delta^{-1}]$$

which concludes the proof.

□

1.2.3 Induced maps

In this paragraph we explain how a continuous map induces a group homomorphism at the level of fundamental groups.

Let (X, x) and (Y, y) two *pointed* topological spaces (which just means that we have a distinguished point for both). We assume that they are path-connected, and let $f : (X, x) \longrightarrow (Y, y)$ a continuous map that maps x to y. f thus maps any loop based at x onto a loop based at y and maps any homotopy with fixed end points at x to a homotopy with fixed end points at y. This shows that if γ_1 and γ_2 are two homotopic (with fixed end points) loops at x, $f \circ \gamma_1$ and $f \circ \gamma_2$ are two homotopic (with fixed end points) loops at y. Altogether, it proves that f induces a map

$$f_* \; := \begin{array}{ccc} \pi_1(X, x) & \longrightarrow & \pi_1(Y, y) \\ [\gamma] & \longmapsto & [f \circ \gamma] \end{array}.$$

Proposition 16. *For any topological spaces as above and a continuous map such that $f(x) = y$, the map f_* defined above is a group homomorphism.*

Proof. This is just a consequence of the fact that $f \circ (\gamma_1 * \gamma_2)$ is equal to $(f \circ \gamma_1) \circ (f \circ \gamma_2)$, and of the fact that f maps the constant path equal to x to the constant path equal to y. □

1.3 Covering spaces

This paragraph covers the material discussed in Lectures 6 and 7.
In this paragraph, we introduce the notion of *covering space*, which is a generalisation of the map $\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ that we have seen in the section about the torus. We will then be relating this notion to the *fundamental group*.

1.3.1 Definition

Definition 5 (Covering map)**.** *Let $\tilde{X}$ and X be topological spaces. A continuous map $\pi : \tilde{X} \longrightarrow X$ is called a covering map if X has a covering by open sets $(U_\alpha)_{\alpha \in A}$ such that*

- *$\forall \alpha \in A$, $\pi^{-1}(U_\alpha)$ is a disjoint union of open sets $\bigcup_{\beta \in B} \tilde{U}_\alpha^\beta$;*
- *$\forall \beta$, π restricted to $\tilde{U}_\alpha^\beta$ is a homeomorphism onto U_α.*

The space $\tilde{X}$ is called a *covering space* of X and the space X is called the *base space*.

Example The map $\begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ z & \longmapsto & z^n \end{array}$ is a covering map.

Exercise 14. *Find a map that is a local homeomorphism but NOT a covering map.*

Proposition 17 (Degree)**.** *Let $\pi : \tilde{X} \longrightarrow X$ be a covering map. Assume X is connected. Then*

$$s := x \mapsto \mathrm{Card}(\pi^{-1}(\{x\}))$$

*is constant. The constant value of s is called the **degree** of π.*

Exercise 15. *Show that a covering map of degree 1 is a homeomorphism.*

Proof. Let x be an arbitrary point in X and let $(U_\alpha)_{\alpha \in A}$ the open cover of X of the definition of a covering map. There exists α_0 such that $x \in U_{\alpha_0}$. Recall that $\pi^{-1}(U_{\alpha_0}))$ is a disjoint union of open sets mapped homeomorphically (in particular bijectively) to U_{α_0}.
Thus for any $y \in U_{\alpha_0}$, there is exactly on $\tilde{y} \in \tilde{X}$ such that $\pi(\tilde{y}) = y$ in each of these open sets. This show that any $y \in U_{\alpha_0}$ has exactly as many pre-images by π as there are open sets mapped homeomorphically to U_{α_0}. In particular, s is constant on U_α for every α. A locally constant function on a connected space is constant, which terminates the proof.

□

1.3.2 Lifting property

A very important property of covering spaces is that a lot of things that happen in the base can be "lifted" up to the covering space. We start with a Proposition which is a straightforward generalisation of Proposition 9 for the torus.

Proposition 18 (Lifts of paths)**.** *Let $\gamma : [0,1] \longrightarrow X$ be a continuous path such that $\gamma(0) = x$. For any $\tilde{x} \in \tilde{X}$ such that $\pi(\tilde{x}) = x$, there exists a unique path $\widetilde{\gamma} : [0,1] \longrightarrow \tilde{X}$ such that*

- $\widetilde{\gamma}(0) = \tilde{x}$;
- $\pi \circ \widetilde{\gamma} = \gamma$.

This Proposition is just a consequence of the more general

Proposition 19 (Lifts of families of maps)**.** *Let Y be a topological space and let*

$$\begin{array}{cccc} f & : & [0,1] \times Y & \longrightarrow & X \\ & & (t,y) & \longmapsto & f_t(y) \end{array}$$

*be a continuous map. Assume there exists a continuous $\tilde{g} : Y \longrightarrow \tilde{X}$ lifting f_0 (that is such that $f_0 = \pi \circ \tilde{g}$. Then there exists a **unique** continuous $\tilde{f}$ lifting f, i.e.*

$$\begin{array}{cccc} \tilde{f} & : & [0,1] \times Y & \longrightarrow & \tilde{X} \\ & & (t,y) & \longmapsto & \tilde{f}_t(y) \end{array}$$

such that

- $\tilde{f}_0 = \tilde{g}$;
- $f = \pi \circ \tilde{f}$.

Remark 20. This Proposition is the one that we needed to formally complete the proof of Proposition 10, applied to the the torus $\mathbb{T}^2$ and its covering space $\mathbb{R}^2$.

Proof. $\pi : \tilde{X} \longrightarrow X$ is a covering map. Recall that it implies the existence of an open cover of X, $(U_\alpha)_{\alpha \in A}$ such that

- $\forall \alpha \in A$, $\pi^{-1}(U_\alpha)$ is a disjoint union of open sets $\bigcup_{\beta \in B} \tilde{U}_\alpha^\beta$;
- $\forall \beta$, π restricted to $\tilde{U}_\alpha^\beta$ is a homeomorphism.

Consider any open set N in Y such that for any $t \in [0, 1]$ there exists an open interval I_t containing t such that $f(N \times I_t) \subset U_\alpha$ for some α. Note for later that for any $y \in Y$, we can find such a N containing x, by continuity of f.
Given such an N we proceed further by finding a finite subdivision of $[0, 1]$, $t_0 = 0 < t_1 < \cdot < t_N < t_{n+1} = 1$ such that for all k, $f(N \times [t_k, t_{k+1}]) \subset U_\alpha$ for some α (we used the compactness of $[0, 1]$).

We start constructing $\tilde{f}$ on $N \times [0, t_1]$. Let $\tilde{U}_\alpha$ be the unique pre-image of U_α such that $\tilde{g}(N \times \{0\}) \subset \tilde{U}_\alpha$ (this can be achieved by restricting N further, by continuity of $\tilde{g}$). Let φ_0 be the inverse of π restricted to $\tilde{U}_\alpha$ (which is a homeomorphism by definition). Set $\tilde{f}$ on $N \times [0, t_1]$ to be $\varphi_0 \circ f$ (which makes sense as by definition $f(N \times [0, t_1]) \subset U_\alpha$).

Assume now that we have constructed a continuous $\tilde{f}$ on $N \times [0, t_k])$ lifting f and agreeing with $\tilde{g}$ on $N \times \{0\}$. We carry on by considering the unique $\tilde{U}_{\alpha_k}$ such that $\tilde{f}(N \times \{t_k\})$ belongs to (as $f(N \times \{t_k\}) \subset$ a certain U_{α_k}). Let φ_k be the inverse of π restricted to $\tilde{U}_{\alpha_k}$. We extend $\tilde{f}$ to $N \times [t_k, t_{k+1}]$ by setting $\tilde{f} = \varphi_k \circ f$. Since this extension agrees with the previous definition of $\tilde{f}$ on $N \times t_k$ and that $\varphi_k \circ f$, the $\tilde{f}$ newly defined is continuous.

This process can be carried out until we get a continuous lift $\tilde{f}$ on $N \times [0, 1]$. We now show that this lift is unique. Assume that we have $\bar{f}$ also lifting f on $N \times [0, 1]$. The set of points where $\tilde{f}$ and $\bar{f}$ agree is closed. But by the lifting property it is also open : if y is such that $\tilde{x} = \tilde{f}(y) = \bar{f}(y)$ let U_α be such that $\pi(\tilde{x}) \in U_\alpha$ and $\tilde{U}_\alpha$ be such that $\tilde{x} \in \tilde{U}_\alpha$. Let V be a neighbourood of y such that $\tilde{f}(V), \bar{f}(V) \subset \tilde{U}_\alpha$. If φ is the inverse of the restriction of π to $\tilde{U}_\alpha$, by continuity of $\tilde{f}$ and $\bar{f}$ we have to have $\tilde{f} = \bar{f} = \varphi \circ f$ on V. The set of points where $\tilde{f}$ and $\bar{f}$ agree is thus open and closed. Since they also agree on N, by a connectedness argument, they agree.

We now define $\tilde{f}$ by covering Y by open sets N such as before. By uniqueness, whenever two such Ns agree, the lift obtained previously must agree on their intersection. So it makes sense to define $\tilde{f}$ by choosing arbitrary N covering Y and take the lift that we have built. Such a map is unique, and continuous as it is continuous on all (open) sets of the form $N \times [0, 1]$. $\square$

Note that Proposition 18 is the particular case where Y is a point.

1.3.3 Covering maps and fundamental group

We show in this paragraph how to associate to a covering map $\pi : \tilde{X} \longrightarrow X$ a subgroup of the fundamental group of X.

Proposition 21. *Let $\pi : \tilde{X} \longrightarrow X$ a covering map, and $\tilde{x_0} \in \tilde{X}$ such that $\pi(\tilde{x_0}) = x_0$. The induced map*

$$\pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$

is injective.

Proof. Assume a class $[\tilde{\gamma}] \in \pi_1(\tilde{X}, \tilde{x}_0)$ is such that $\pi_*([\tilde{\gamma}])$ is trivial. Consider a representative $\tilde{\gamma}$. It means that $\pi \circ \tilde{\gamma}$ is homotopic to a constant path via $H(s,t)$ such that for every $H(s,0) = H(s,1) = x_0$. By Proposition 9, there exists a unique lift $\tilde{H}$ of H such that $\tilde{H}(0,t) = \tilde{\gamma}$. Furthermore, $\tilde{H}(t,0)$ and $\tilde{H}(t,1)$ are continuous and lift the constant path x_0. Since $H(0,0) = \tilde{H}(0,1) = \tilde{x}_0$ by uniqueness we have $\tilde{H}(t,0) = \tilde{H}(t,1) = \tilde{x}_0$ for all t. Same applies for $\tilde{H}(1,\cdot)$. $\tilde{H}$ is therefore a homotopy with fixed end points between $\tilde{\gamma}$ and the constant path at $\tilde{x_0}$. This proves that $[\tilde{\gamma}] = 0$. The kernel of π_* is thus trivial.

□

1.4 Universal cover

This paragraph covers the material discussed in Lecture 8. In this paragraph, X is assumed to be a *path-connected, locally path-connected and locally simply-connected*[2] topological space.

Definition 6. *A space $\tilde{X}$ is a* ***universal cover*** *of X if $\tilde{X}$ is simply connected.*

Proposition 22. *A universal cover of X, if it exists, is unique. By that we mean that if $\pi_1 : \tilde{X}_1 \to X$ and $\pi_2 : \tilde{X}_2 \to X$ are two universal covers of X there exists a homeomorphism $\varphi : \tilde{X}_1 \longrightarrow \tilde{X}_2$ such that the following diagram commutes*

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\varphi} & \tilde{X}_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{\mathrm{Id}} & X \end{array}$$

This Proposition does not settle the existence of a universal cover. At a theoretical level we have the following Theorem.

Theorem 23. *Any space X that is path-connected, locally path-connected and locally simply-connected has a universal cover.*

The reason we do not linger on its proof is because in practise we will not be using this existence theorem much. In most situations, the universal cover of a space X will be easy to find (like $\mathbb{R} \to S^1$ or $S^n \longrightarrow \mathbb{RP}^n$). At any rate, knowing the existence of the universal cover but not *who* it is is of no particular use.

In order to prove the uniqueness of the universal cover, we prove a general lemma that we will be using in several place.

[2] A space is *locally something* if every point has a basis of neighbourhoods that are all *something*.

Lemma 24. *Let Y be a simply-connected and locally path-connected space and let $\pi : \bar{X} \to X$ be a covering map. Consider $f : Y \longrightarrow X$. Then for any y, x and $\bar{x}$ such that $f(y) = \pi(\bar{x}) = x$, there exists a unique continuous lift*

$$\bar{f} : Y \longrightarrow \bar{X}$$

such that $\bar{f}(y) = \bar{x}$ and $\pi \circ \bar{f} = f$.

Proof. **Definition of $\bar{f}$.** Let y' be a point in Y and consider an arbitrary path γ from y to y'. Consider $\bar{\gamma}$ any lift to $\bar{X}$ of $f \circ \gamma$ starting at $\bar{x}$. We define

$$\bar{f}(y') := \bar{\gamma}(1).$$

To see that it is well-defined, we show that it does not depend on the choice of γ. Any other path γ' from y to y' is homotopic to γ with fixed end points (as Y is simply-connected). The image of an homotopy between $f \circ \gamma$ and $f \circ \gamma'$ can be lifted to a homotopy with fixed end points between lifts to $\bar{X}$ based at $\bar{x}$. This shows that $\bar{f}$ is well-defined.

Continuity of $\bar{f}$. Let U be an open set of X containing $f(y')$ and such that $\pi^{-1}(U)$ is a union of homeomorphic copies of U to which π restrict to a homeomorphism onto U. Let $\bar{U}$ be the copy of U in $\bar{X}$ $\bar{f}(y')$ belongs to. Let y' be a point in Y and let V be a small neighbourhood of y' such that V is path-connected and $f(V)$ is contained in U. We show that $\bar{f}^{-1}\bar{U}$ contains V. Indeed, any path γ from y to y' can be concatenated to a path γ' in V to reach any point y'' in V. By definition of $\bar{f}$, $\bar{f}(y'')$ is the lift of $f \circ \gamma'$ to $\bar{X}$ based at $\bar{x}$. Since $\bar{f}(y') \in \bar{U}$, the lift of the part δ of γ' from y' to y'' stays in $\bar{U}$ as $f \circ \delta$ stays in U. This implies that for y'' in V, $\bar{f}(y'') \in \bar{U}$. Since the $\bar{U}$ form a basis of open sets of $\bar{X}$, $\bar{f}$ is continuous.

Uniqueness of $\bar{f}$. A $\bar{f}$ satisfying the hypothesis of the Proposition in particular satisfies that for any γ based at y, $\bar{f}\gamma$ is a path based at $\bar{x}$ lifting $f \circ \gamma$. By uniqueness of the lift, $\bar{f}$ must be unique.

□

Proof of Proposition 22. It suffices to take the lift given by Lemma 24 of π_1 to $\tilde{X}_2$ mapping $\tilde{x}_1$ to $\tilde{x}_2$ to obtain the map φ. Because $\pi_2 \circ \varphi = \pi_1$, φ is a local homeomorphism. Applying the same reasoning to lift π_2 to $\tilde{X}_1$ mapping $\tilde{x}_2$ to $\tilde{x}_1$, we obtain a map $\psi : \tilde{X}_2 \longrightarrow \tilde{X}_1$ such that $\psi \circ \varphi = \mathrm{Id}_{\tilde{X}_1}$ and $\varphi \circ \psi = \mathrm{Id}_{\tilde{X}_2}$. ψ is a continuous inverse of φ which is thus a homeomorphism. □

We derive another important consequence of Lemma 24, which is one step towards the classification of covering space.

Proposition 25 (Factorisation of covering maps)**.** *Let $\pi : (\tilde{X}, \tilde{x}) \longrightarrow (X, x)$ be the universal cover of X and let $p : (\bar{X}, \bar{x}) \longrightarrow (X, x)$ a covering map. There exists a unique covering map $f : (\tilde{X}, \tilde{x}) \longrightarrow (\bar{X}, \bar{x})$ such that $p \circ f = \pi$.*

Proof. Straightforward application of Lemma 24.

□

1.5 Group actions and quotients

This paragraph covers the material discussed in Lecture 9.

In this section we introduce a construction which generalises the identification of the torus by an action of the group $\mathbb{Z}^2$ on $\mathbb{R}^2$. This construction is can be summarised the following way: *forming the quotient of a space by a "nice" action of a group*. We then explain the connection between this construction and covering spaces and the fundamental group.

Definition 7 (Group action)**.** *Let Γ be a group and X a topological space. A (topological) action of Γ on X is a group homomorphism*

$$\Gamma \longrightarrow \mathrm{Homeo}(X).$$

Alternatively, a group action can be defined as a map

$$\begin{array}{ccc} (\Gamma, X) & \longrightarrow & X \\ (g, x) & \longmapsto & g \cdot x \end{array}$$

such that for all $g_1, g_2 \in \Gamma$ and $x \in X$, $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $e \cdot x$ (where $e \in \Gamma$ is the neutral element).

Definition 8 (Nice action)**.** *An action of a group Γ on a topological space X is said to be nice (of satisfies Axiom N) if*

For any $x \in X$ there exists U neighbourhood of x such that for all $\gamma \in \Gamma$ different from the identity, $\gamma \cdot U \cap U = \emptyset$.

Remark 26. For a group action Γ on a space X, we will often think of $\gamma \in \Gamma$ as an element of $\mathrm{Homeo}(X)$. For instance, we will say the *the identity* thinking of the action of the neutral element in Γ, and we will write $\gamma \cdot x \in X$ meaning the image of x by the image of γ via the group homomorphism $\Gamma \longrightarrow \mathrm{Homeo}(X)$.

Let Γ be a group acting of a topological space X. We introduce the equivalence relation $x \sim y \Leftrightarrow \exists \gamma \in \Gamma$ such that $y = \gamma \cdot x$. The reader can verify that such a relation is indeed an equivalence relation. In the sequel we will use the notation

$$X/\Gamma$$

for $X/_\sim$. The important result that we will be using to connect group actions to covering maps is the following

Proposition 27. *Let X be a topological space and let Γ be a group acting on X nicely (that is satisfying Axiom N above). Then the projection*

$$\pi : X \longrightarrow X/\Gamma$$

is a covering map.

Proof. For any $x \in X$, let U_x be the neighbourhood of x given by the definition of a nice action. By definition of the nice action $\pi^{-1}(\pi(U_x)) = \bigcup_{\gamma \in \Gamma} \gamma(U)$ (where the union is disjoint) and for all γ, $\pi(\gamma(U)) = \pi(U)$, and $\pi_{\gamma U}$ is a bijection. We show that $\pi(U_x)$ is open. This is just because $\pi^{-1}(\pi(U_x)) = \bigcup_{\gamma \in \Gamma} \gamma(U)$ is a union of open sets, as the γs are homeomorphisms. This argument extends to any open set contained in U_x, which shows that the inverse map of π restricted to U_x is continuous. Thus π is a homeomorphism restricted to U_x. Since the U_xs cover X, π is a covering map.

□

From this result we derive the following important consequence.

Proposition 28. *Let X be a simply-connected space and Γ be a group acting nicely on X. Then*

1. *X is the universal cover of X/Γ via the covering map $\pi : X \longrightarrow X/\Gamma$;*
2. *for any $[x] \in X/\Gamma$, the fundamental group $\pi_1(X/\Gamma, [x])$ is isomorphic to Γ.*

Proof. The first point is just a consequence of Proposition 27 combined with the uniqueness of the universal cover (when it exists).

Let us focus on the second point. Consider a path $\delta(\gamma)$ based at $x \in X$ to $\gamma.x$. It projects in X/Γ onto a closed path based at $[x]$. Since X is simply-connected, any two such paths are homotopic with fixed end points, and so are their projection in X/Γ. We can therefore define a map

$$\varphi : \Gamma \longrightarrow \pi_1(X/\Gamma, [x]).$$

One can check that this map is a group homomorphism (sends the identity onto a path homotopic to a constant, and we can find paths from x to $\gamma_1\gamma_2 x$ through $\gamma_2 x$, to be broken into two paths which respectively project to a path representing $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$.

This map is injective. Indeed, as for any x and $\gamma \neq Id$, $\gamma \cdot x \neq x$ (the action is without fixed point, by the definition of a nice action), a representative of $\varphi(\gamma)$ lifts to X to a a path with different end points. Since π is a covering map, $\varphi(\gamma)$ is non-trivial.

This map is surjective. Any non-trivial path in X/Γ based at $[x]$ can be lifted to a path based at x. Since the end point of the lift also maps to $[x]$ it must be of the form $\gamma \cdot x$ for a certain γ. This proves the surjectivity.

□

We conclude this section by a converse to Proposition 28. We construct, for any reasonable space X, a nice action of $\pi_1(X, x)$ on the universal cover $\tilde{X}$ so that $\tilde{X}/\pi_1(X, x) = X$. Take an arbitrary $x \in X$ and let γ be path based at x. We consider a lift of γ to $\tilde{X}$ starting at $\tilde{x}$ and ending in $\gamma\tilde{x}$.

Let $\rho(\gamma)$ be the lift of the identity of X (given by Lemma 24) which maps $\tilde{x}$ to $\gamma\tilde{x}$. One easily checks that $\rho(\gamma)$ does not depend on a representative of the class of γ in $\pi_1(X, x)$. We have the following Lemma.

Lemma 29. *Let γ_1 and γ_2 be two elements of $\pi_1(X,x)$. Any path from $\rho(\gamma_1)(\tilde{x})$ to $\rho(\gamma_2)(\rho(\gamma_1)((\tilde{x}))$ projects in X to a curve representing $\gamma_1\gamma_2\gamma_1^{-1}$.*

Proof. (We strongly recommend the reader to draw a picture).

- Consider a path from δ_1 from $\tilde{x}$ to $\rho(\gamma_1)(\tilde{x})$. δ_1 projects to a path representing γ_1.
- The image of δ_1 by the action of $\rho(\gamma_2)$ of δ_1 is a path from $\rho(\gamma_2)(\tilde{x})$ to $\rho(\gamma_2)(\rho(\gamma_1)(\tilde{x}))$. $\rho(\gamma_1)\circ\delta_1$ projects to a path representing γ_1.
- Let δ_2 be a path from $\tilde{x}$ to $\rho(\gamma_2)(\tilde{x})$. δ_2 projects to a path representing γ_1.
- Finally let μ be a path from $\rho(\gamma_1)(\tilde{x})$ to $\rho(\gamma_2)(\rho(\gamma_1)(\tilde{x}))$.

Since $\tilde{X}$ is simply connected, μ is homotopic with fixed end points to $\delta_1^{-1} * \delta_2 * \rho(\gamma_2)(\delta_1)$. This implies that the projection of μ in X represents $\gamma_1\gamma_2\gamma_1^{-1}$.

□

Proposition 30. *1. $\rho : \pi_1(X,x) \longrightarrow \mathrm{Homeo}(\tilde{X})$ is a group homomrophism.*

2. If X is locally connected the action of $\pi_1(X,x)$ thus defined is nice.

Proof. ρ **is a group homomorphism.** Let γ_1 and γ_2 two paths based at x. Consider $\rho(\gamma_2)\circ\rho(\gamma_1)(\tilde{x}) = \rho(\gamma_2)(\rho(\gamma_1)(\tilde{x}))$. $(\rho(\gamma_1)(\tilde{x}))$ is the end point of a lift of γ_1 based at $\tilde{x}$, and $\rho(\gamma_2)(\rho(\gamma_1)(\tilde{x}))$ of a path from $(\rho(\gamma_1)(\tilde{x}))$. By Lemma 29, the latter path lifts a the class $\gamma_1\gamma_2\gamma_1^{-1}$. Concatenating this two lifts, we get that $\rho(\gamma_2)\circ\rho(\gamma_1)(\tilde{x})$ is the end point of a lift of $\gamma_1\gamma_2\gamma_1^{-1} * \gamma_1 = \gamma_1\gamma_2$ based at $\tilde{x}$. Thus

$$\rho(\gamma_2)\circ\rho(\gamma_1)(\tilde{x}) = \rho(\gamma_2 * \gamma_1)(\tilde{x}).$$

$\rho(\gamma_2)\circ\rho(\gamma_1)$ and $\rho(\gamma_2 * \gamma_1)$ are two lifts of the identity which agree at one point, they are therefore equal. Finally, $\rho(e)$ is a lift of the identity agreeing with $Id_{\tilde{X}}$ at one point, it is therefore $Id_{\tilde{X}}$. This shows that ρ is a group homomorphism.

The action of $\pi_1(X,x)$ via ρ is nice. Let $x \in X$ and let U_x be a neighbourhood of x in X given by the definition of covering space. Then $\pi^{-1}(U)$ is a disjoint collection of open sets each mapping homeomorphically to U via π. Let $\gamma \neq \mathrm{Id}$, and let $\tilde{x}$ an element in the pre-image of U and let $U_{\tilde{x}}$ the open set of $\pi^{-1}(U)$ $\tilde{x}$ belongs to. $\gamma\tilde{x}$ projects onto x as well and therefore belongs to a different copy of U which we denote by $U_{\gamma\tilde{x}}$. Now the map

$$\begin{array}{ccc} U_{\gamma\tilde{x}} & \longrightarrow & U_{\tilde{x}} \\ x & \longmapsto & (\pi_{U_{\tilde{x}}})^{-1}\circ\pi(x) \end{array}$$

is a lift of Id_U and agrees with $\rho(\gamma)$ at $\tilde{x}$. If U had been chosen connected to begin with, this map would have to agree with $\rho(\gamma)$ (as they both lift Id_U and agree at a point). Since U_{tildex} and $U_{\gamma\tilde{x}}$ are disjoint, this shows that the action of $\pi_1(X,x)$ is nice.

□

1.6 Classification of covering maps

This paragraph covers the material discussed in Lecture 10.

We consider pointed covering maps, that is covering maps

$$p : (\bar{X}, \bar{x}) \longrightarrow (X, x)$$

where $p(\bar{x}) = x$. We have seen that the map

$$p_* : \pi_1(\bar{X}, \bar{x}) \longrightarrow \pi_1(X, x)$$

is injective. We have this way associated to any such covering map a subgroup of $\pi_1(X, x)$ isomorphic to $\pi_1(\bar{X}, \bar{x})$. Our aim in this paragraph is to show that this correspondence is essentially one-to-one.

1.6.1 Covering map associated to a subgroup.

We assume that X is **path-connected**, **locally path-connected** and **locally simply-connected** to ensure that X has a *universal cover* $\tilde{X}$. In particular there is a nice action of $\pi_1(X, x)$ on $\tilde{X}$ such that

$$X = \tilde{X}/\pi_1(X, x).$$

Let $\Gamma < \pi_1(X, x)$ be a subgroup of $\pi_1(X, x)$. The restriction of the action of $\pi_1(X, x)$ to Γ is still a nice action and we can therefore consider the quotient space

$$X_\Gamma = \tilde{X}/\Gamma.$$

It is easily checked that the natural map

$$p : X_\Gamma = \tilde{X}/\Gamma \longrightarrow \tilde{X}/\pi_1(X, x) = X$$

is a covering map such that $p_*(\pi_1(X_\Gamma, x_g)) = \Gamma$ (where x_g is any point in $p^{-1}(\{x\})$).

1.6.2 Uniqueness of the covering map

We have the following important Proposition which proves the essential uniqueness of the covering map with associated subgroup Γ.

Proposition 31. *Let $\bar{p} : (\bar{X}, \bar{x}) \longrightarrow (X, x)$ be a covering map such that $\bar{p}(\pi_1(\bar{X}, \bar{x})) = \Gamma$. Then there exists φ such that the following diagram commutes*

$$\begin{array}{ccc} (X_\Gamma, x_g) & \xrightarrow{\varphi} & (\bar{X}, \bar{x}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ (X, x) & \xrightarrow{\mathrm{Id}} & (X, x) \end{array} .$$

Proof. The construction of φ follows the usual recipe. By definition, $X_\Gamma = \tilde{X}/\Gamma$ where $\tilde{X}$ is the universal cover of X. We consider the lift of $\pi : \tilde{X} \to X$ to $\bar{X}$ that maps any lift $\tilde{x}$ of x_g to $\bar{x}$, which we call $\tilde{\varphi}$.

$\tilde{\varphi}$ **is Γ-invariant.** Consider $\gamma \in \Gamma$, a path δ from $\tilde{x}$ to $\gamma\tilde{x}$. The curve $\tilde{\varphi} \circ \delta$ represents the class $\gamma \in \pi_1(X, x)$. Since $\gamma \in \bar{p}(\pi_1(\bar{X}, \bar{x}))$, it lifts to a closed curve in $\bar{X}$ based at $\bar{x}$, which implies $\tilde{\varphi}(\tilde{x}) = \tilde{\varphi}(\gamma\tilde{x})$.

Note that $\tilde{\varphi} \circ \gamma$ is also a (continuous) lift of π to $\bar{X}$ since γ is a lift of the identity to $\tilde{X}$. But we have just shown that $\tilde{\varphi} \circ \gamma$ and $\tilde{\varphi}$ agree on $\tilde{x}$, we obtain

$$\tilde{\varphi} \circ \gamma = \tilde{\varphi}.$$

Thus $\tilde{\varphi}$ induces a continuous

$$\varphi : (X_\gamma, x_g) \longrightarrow (\bar{X}, \bar{x})$$

lifting the identity on (X, x).

φ **is a homeomorphism.** By construction φ is surjective and continuous. The exact same construction starting with a lift of p (from $\bar{X}$ to X_Γ) constructs a continuous inverse for φ thus proving that it is a homeomorphism. □

1.7 Van Kampen Theorem

This paragraph covers the material discussed in Lecture 11.

1.7.1 Free groups and presentations

Let $a_1, \cdots, a_n$ be n letters and consider $\mathcal{A}$ the alphabet on $2n$ letters $a_1, \cdots, a_n, a_1^{-1}, \cdots, a_n^{-1}$. Let w be a word with letters in $\mathcal{A}_n$. We say that w is *reduced* is no two consecutive letters are of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ for some i. A w word that is not reduced can be put in its reduced form $r(w)$ by inductively erasing pairs of consecutive letters of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$.

Definition 9. *The free group with n generators $\mathbb{F}_n$ is the group formed of reduced words on the alphabet $\mathcal{A}_n$ with the product*

$$w_1 * w_2 = r(w_1 w_2).$$

In this case, the elements $a_1, \cdots a_n$ generate $\mathbb{F}_n$ as a group and we thus note

$$\mathbb{F}_n = \langle a_1, \cdots a_n \rangle.$$

Consider a group G that is finitely generated, with generators $g_1, \cdots, g_n$. The map

$$\varphi \quad := \quad \begin{array}{ccc} \mathbb{F}_n & \longrightarrow & G \\ w(a_1, \cdots, a_n, a_1^{-1}, \cdots, a_n^{-1}) & \longmapsto & w(g_1, \cdots, g_n, g_1^{-1}, \cdots, g_n^{-1}) \end{array}$$

where w is any word on $2n$ letters, is a surjective group homomorphism. We therefore realise G as

$$G = \mathbb{F}_n/\mathrm{Ker}(\varphi).$$

We have obtained the following

Any finitely generated group is isomorphic to a quotient of a free group.

In the other direction, we explain a natural construction of groups from the free group. Consider $(r_i)_{i\in I}$ a family of reduced words in $a_1, \cdots, a_n, a_1^{-1}, \cdots, a_n^{-1}$. Let $\langle\langle r_i || \ i \in I\rangle\rangle$ be the **normal** subgroup generated by the r_is (by that we mean the smallest normal subgroup containing all the r_is). Beware, $\langle\langle r_i || \ i \in I\rangle\rangle$ is NOT (in general) the subgroup generated by the r_is but the subgroup generated by all conjugates of r_is in $\mathbb{F}_n$. We set the following notation

$$\langle a_1, \cdots, a_n \mid r_i(a_1, \cdots, a_n) = 1, \ i \in I\rangle := \mathbb{F}_n/\langle\langle r_i || \ i \in I\rangle\rangle.$$

One should think of this group as "the group generated by the a_is where the a_is satisfy the relations r_i, $i \in I$. Such a realisation of a group G (with generators and relations) is called a *presentation* of G.

Exercise 16. *Show that $\mathbb{Z}^2$ is isomorphic to $\langle a_1, a_2 \mid a_1 a_2 a_1^{-1} a_2^{-1} = 1\rangle$.*

1.7.2 Almagamated products

Let G_1 and G_2 be two finitely generated groups with presentations
$G_1 = \langle a_1, \cdots, a_n \mid r_i(a_1, \cdots, a_n) = 1, \ i \in I\rangle$ and $G_2 = \langle b_1, \cdots, b_m \mid s_j(b_1, \cdots, b_m) = 1, \ j \in J\rangle$.

Definition 10 (Free product)**.** *The free product of G_1 and G_2 is the group*

$$G_1 * G_2 = \langle a_1, \cdots, a_n, b_1, \cdots, b_m \mid r_i(a_1, \cdots, a_n) = 1, \ s_j(b_1, \cdots, b_m) = 1, \ i \in I, \ j \in J\rangle.$$

In other words, a free product between two groups is just formal products between elements of the two groups without any extra other relations between these products other than the obvious ones. We now define the *amalgamated product* between two groups, which formalises the idea of glueing two groups together along two of their subgroups. Formally, let H be a group and let $h_1 : H \to G_1$, $h_2 : H \to G_2$ be two group morphisms.

Definition 11 (Amalgamated product)**.** *The amalgamated product of G_1 and G_2 along H, which we denote by $G_1 *_H G_1$, is the quotient of $G_1 * G_2$ by the normal subgroup generated by elements of the form $h_1(h) \cdot h_2(h)^{-1}$, $h \in H$.*

Exercise 17. *Show that $\mathbb{F}_n * \mathbb{F}_m = \mathbb{F}_{n+m}$.*

1.7.3 Van Kampen theorem

Now that we have introduced some elements of group theory, we state and partially prove a theorem allowing us to compute fundamental groups of spaces if we already know the fundamental group of some carefully chosen subsets.

Theorem 32 (Van Kampen Theorem)**.** *Let X be a path-connected space and let A and B two-path connected open subsets of X such that $A \cap B$ is also path-connected and $X = A \cup B$. Consider x_0 in $A \cap B$ and let $i_A : A \cap B \to A$ and $i_B : A \cap B \to B$ the respective injections of $A \cap B$ in A and B. Then*

$$\pi_1(X, x_0) \simeq \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$

where the amalgamated product is taken with respect to the induced maps of the two injections $(i_A)_*$ and $(i_B)_*$.

Proof. First, note that the injections $A \hookrightarrow X$ and $B \hookrightarrow X$ define two group homomorphisms $\pi_1(A, x_0) \longmapsto \pi_1(X, x_0)$ and $\pi_1(B, x_0) \longmapsto \pi_1(X, x_0)$ which extend to a group homomorphism

$$\varphi : \pi_1(A, x_0) * \pi_1(B, x_0) \longrightarrow \pi_1(X, x_0).$$

Now, two elements $i_A(h) \subset \pi_1(A, x_0)$ and $i_B(h) \subset \pi_1(B, x_0)$ do map to the same elements via φ as $\varphi(ß_A)_*$ and $\varphi(ß_B)_*$ are both the induced map of the inclusion $A \cap B \overset{X}{\hookrightarrow}$. Thus any element of the form $(i_A)_*(h)(i_B)_*(h)^{-1}$ is in the kernel of φ. This implies that φ induces an homomorphism

$$\phi := \pi_1(A, x_0) *_{\pi_1(A \cap B), x_0} \pi_1(B, x_0).$$

Remains to show that ϕ thus defines an isomorphism. We will only show the surjectivity.

Surjectivity. Let $\gamma : [0, 1] \longrightarrow X$ based at x_0. Since $X = A \cup B$ and A and B are open, for all $t \in [0, 1]$ there exists a connected open (in $[0, 1]$) interval I_t such that $\gamma(I_t)$ is in A or B. By compactness we can cover $[0, 1]$ by a finite number of such I_ts and find a finite subdivision $t_0 = 0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$ so that for all k, $\gamma([t_k, t_{k+1}]) \subset A$ or B and $\gamma(t_k) \in A \cap B$. Set $x_k = \gamma(t_k)$ and for all k choose an arbitrary path δ_k in $A \cap B$ from x_0 to x_k. Set $\gamma_k = \gamma_{|[t_k, t_{k+1}]}$. We can thus write

$$\gamma = \gamma_1 \delta_1^{-1} \delta_1 \gamma_2 \delta_2^{-1} \delta_2 \cdots \delta_{n-1}^{-1} \delta_{n-1} \gamma_n.$$

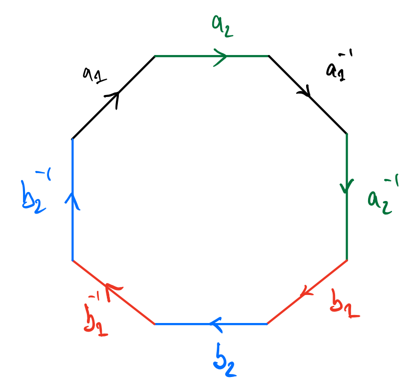
For all k, $\delta_k \gamma_{k+1} \delta_{k+1}^{-1}$ (with the convention $\delta_{n+1} = \delta_0$ is the constant path at x_0) is a closed path in either A or B. Group term accordingly, we have written γ as a product of closed paths based at x_0 included either in A or B. This implies that φ is surjective.

Injectivity. Injectivity follows the same line of thought but is slightly more involved. The goal is to show that a map in the kernel of φ is in the normal subgroup generated by elements of the form $(i_A)_*(h)(i_B)_*(h)^{-1}$. We omit the detail of the proof of this fact and refer, for instance, to the book by Hatcher.

□

1.8 An application of Van Kampen theorem : the fundamental group of a genus 2 surface

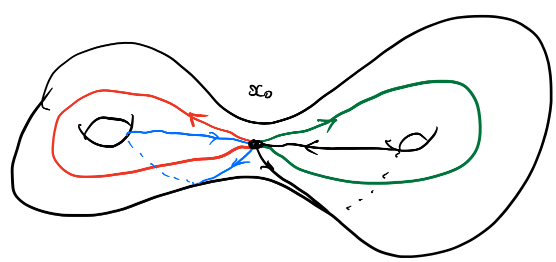
The surface of genus 2 is the quotient space obtained from an octogon P with the identifications of its side as indicated in the Figure below.



We set by definition

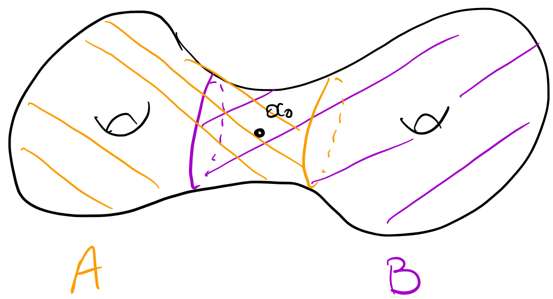
$$\Sigma = P/_\sim.$$

We are going to compute the fundamental group of Σ, using Van Kampen theorem. We consider a point x_0 as in Figure below and closed paths based at x_0, a_1, a_2, b_1 and b_2.



1.8.1 First method : Σ as the union of two holed tori.

We write Σ as a union of two holed tori A and B as in Figure below. Let δ a curve of $A \cap B$, based at x_0 that is *freely* homotopic to the boundary curve of both A and B.



Working out the algebra. We apply Van Kampen theorem to $X = A \cup B$, we obtain that

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0)$$

where the amalgamation is relative to the induced map of the injections $i_A : A \cap B \to A$, $i_B : A \cap B \to B$. Precisely, that means that $\pi_1(X, x_0)$ is the quotient of $\pi_1(A, x_0) * \pi_1(B, x_0)$ by the subgroup

$$H = << (i_A)_*(h) \cdot (i_B)_*(h)^{-1} \mid h \in \pi_1(A \cap B, x_0) >>$$

which the NORMAL subgroup generated by elements of the form $(i_A)_*(h) \cdot (i_B)_*(h)^{-1}$.

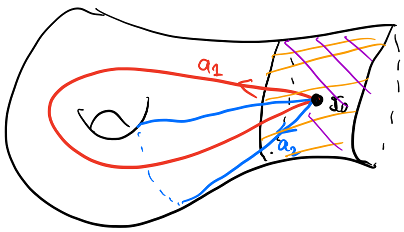
In our case $A \cap B$ is homeomorphic to $S^1 \times (0, 1)$ (which deformation retracts onto S^1), thus $\pi_1(A \cap B, x_0)$ is isomorphic to $\mathbb{Z}$ and is generated by the class of δ. The subgroup H is therefore the normal subgroup generated by $(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1}$. In other words,

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1}$$

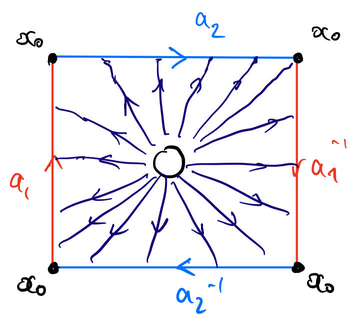
is the only extra relation to be added to $\pi_1(A, x_0) * \pi_1(B, x_0)$ to obtain $\pi_1(X, x_0)$.

That's us done with the algebra. We are now left to computing (that is giving a presentation of) $\pi_1(A, x_0)$, $\pi_1(B, x_0)$ and computing $(i_A)_*([\delta])$ and $(i_B)_*([\delta]))^{-1}$.

Presentation of $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$. We only treat the case of A as B is exactly the same. A is the one holed torus as in Figure below, where the dashed area is $A \cap B$.



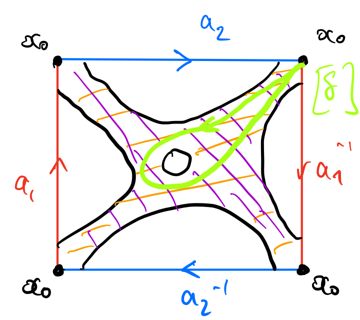
Cutting along a_1 and b_1, we can realise A as the quotient of the following square (where the four vertices project onto x_0).



On this Figure one sees that A deformation retracts on $a_1 \vee b_1 \simeq S^1 \vee S^1$. Thus, $\pi_1(A, x_0)$ is free generated by a_1 and a_2. Same holds for $\pi_1(B, x_0)$, which is freely generated by b_1 and b_2. We have obtained the following Proposition

Proposition 33. *$\pi_1(A, x_0) * \pi_1(B, x_0)$ is a free group, freely generated by a_1, a_2, b_1 and b_2.*

Computation of $(i_A)_*([\delta])$ and $(i_B)_*([\delta])$ We just add δ to the Figure below



Here one sees that δ can be homotoped (in A!) to $a_1a_2a_1^{-1}a_2^{-1}$. This implies that

$$(i_A)_*([\delta]) = a_1a_2a_1^{-1}a_2^{-1}.$$

A similar reasoning shows that δ can be homotoped (in B this time) to $b_2^{-1}b_1^{-1}b_2b_1$ (check that it is consistent with the orientation chosen on the Figure!) to obtain

$$(i_B)_*([\delta]) = b_2^{-1}b_1^{-1}b_2b_1.$$

Therefore

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1} = a_1a_2a_1^{-1}a_2^{-1}b_1b_2b_1^{-1}b_2^{-1}.$$

We use the notation $[a,b] = aba^{-1}b^{-1}$ to write

$$(i_A)_*([\delta]) \cdot ((i_B)_*([\delta]))^{-1} = [a_1,a_2] \cdot [b_1,b_2].$$

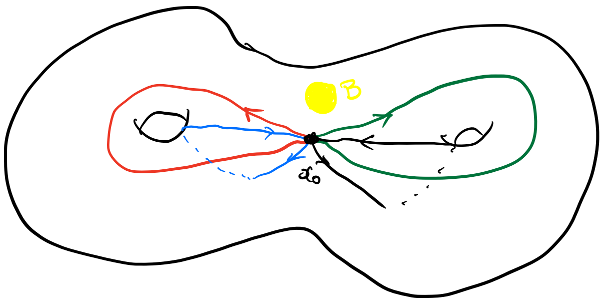
Putting everything together we obtain the following

Theorem 34. *The fundamental group of Σ is isomorphic to*

$$< a_1, a_2, b_1, b_2 \mid [a_1,a_2] \cdot [b_1,b_2] = 1 > .$$

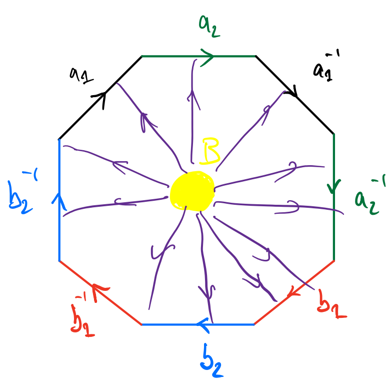
1.8.2 Second method : $\Sigma \setminus \{p\}$ deformation retracts on a wedge of 4 circles.

We sketch the proof of a second method. Take a point p different from x_0 and let B be a little disc around p and let C be a small disk whose closure is contained in the interior of B. Let $A := \Sigma \setminus C$. We put x_0 in $A \cap B$.

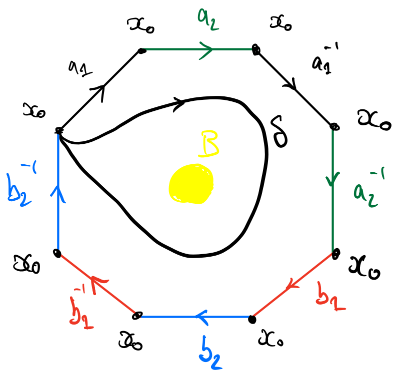


We apply Van Kampen to $X = A \cup B$. We have

- $A \cap B$ is homeomorphic to $S^1 \times (0,1)$ with generator δ as in Figure below.
- $\pi_1(B, x_0)$ is trivial.



We consider A and cut along a_1, a_2, b_1 and b_2 to see A as quotient as Figure below, with all the vertices of the octogon projecting onto x_0 in the quotient.



On can see thus that A deformation retract onto $a_1 \vee a_2 \vee b_1 \vee b_2$ the wedge of four circles. The fundamental group of A is therefore isomorphic to the free group generated by a_1, a_2, b_1 and b_2.

Adding δ in the picture, one show that $(i_A)_*(\delta) = [a_1, a_2] \cdot [b_1, b_2] \in \pi_1(A, x_0)$ (keeping the notations of the first method) and $(i_B)_*(\delta) = 1 \in \pi_1(B, x_0)$. Working out the algebra of Van Kampen yields another proof of Theorem 34.

Chapter 2

Homology

2.1 Δ-complexes and simplicial homology

This paragraph covers the material discussed in Lectures 12 and 13.

We introduce here a class of topological spaces called Δ-*complexes*, which is fairly general (it contains graphs, topological manifolds and virtually all spaces that we have encountered so far) and which are (unlike the arbitrary topological space) amenable to analysis using algebraic techniques to be introduced in the coming sections.

2.1.1 The n-dimensional simplex

We define the *n-dimensional simplex* to be

$$\Delta_n := \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \forall i, x_i \geq 0 \text{ and } \sum x_i = 1\}.$$

For any i, let e_i be the vector whose coordinates are all 0 apart from the i-th which is 1. We use the notation

$$\Delta_n = [e_0, e_1, \cdots, e_n]$$

to mean that Δ_n is the (closed) convex set spanned by the e_is. Note that

1. Δ_n is a closed subset of $\mathbb{R}^{n+1}$;
2. the boundary of Δ_n is a union of $n+1$ copies of Δ_{n-1}, which are exactly the convex sets spanned by all but one of the e_i;

We are now going to consider the simplices $[e_0, e_1, \cdots, e_n]$ and $[e_1, e_0, \cdots, e_n]$ as *different* simplices: there are simplices together with an *orientation* . Formally speaking, we decided that if σ is a permutation, $[e_0, e_1, \cdots, e_n] = [e_{\sigma(0)}, e_{\sigma(1)}, \cdots, e_{\sigma(n)}]$ if and only if the vectors $e_0, e_1, \cdots, e_n$ in this order induce the same orientation as $e_{\sigma(0)}, e_{\sigma(1)}, \cdots, e_{\sigma(n)}$ on the vector space they span. This is equivalent to the signature of σ being equal to 1.
Inductively, one sees that Δ_n contains $\binom{n+1}{k}$ copies of Δ_{n-k} of the form, which we denote by

$$[e_{i_1}, \cdots, e_{i_k}]$$

where the $(i_j)_{1\leq j\leq k}$ are pairwise distinct indices, and which is the convex hull of $\{e_{i_1}, \cdots, e_{i_k}\}$.

2.1.2 Δ-complexes

Let X be a set, together with, for all $k \leq n$, a finite number of maps $\varphi_i^k : \Delta_k \longrightarrow X$, $i \leq m_k$ such that

- for all k and i, φ_i^k is a homeomorphism on the interior of Δ_k;
- for all i, the restriction of φ_i^k to one of the boundary copies $\Delta_{k-1} \subset \Delta_k$ is equal to one of the φ_j^{k-1} for some j;
- $$\bigcup_{k=0}^{n} \bigcup_{i=1}^{m_k} \varphi_i^k(\Delta_k) = X.$$

Such a set X and a set of maps is called a Δ-*complex*.

Topology on X. We endow X with the following topology : a set U is open if and only if for all k and i, $(\varphi_i^k)^{-1}(U)$ is open in Δ_k. In other words, it is the smallest topology making all the (φ_i^k)s continuous.

Remark 35. We have restricted our attention to *finitely generated* Δ-complexes of *finite dimension*. We could have allowed for an infinite number of simplices to be used in the definition; and we could have allowed for the use of simplices of arbitrary large dimension, yielding a slightly more general class of spaces. To simplify the discussion (and because it is already enough for the building of a rich theory) we restrict ourselves to finitely generated ones.

2.1.3 Chains and boundary operators

We now consider such a space X with a Δ-complex structure. Fix $k \leq n$, and for each $i \leq m_k$, introduce the symbol σ_i^k that represents the *oriented* simplex $\varphi_i^k(\Delta_k) \subset X$. We define

$$\mathcal{C}_k(X) := \bigoplus_{i=1}^{m_k} \mathbb{Z} \cdot \sigma_i^k$$

the free abelian group generated by the σ_i^ks. An element in $\mathcal{C}_k(X)$ is just a formal linear combination of simplices with coefficient in $\mathbb{Z}$.

Boundary operator. We denote by $\partial_i \Delta_k$ the facet of Δ_k generated by $e_0, \cdots, e_{i-1}, e_{i+1}, \cdots, e_n$ together with the orientation induced by the natural ordering of the e_is. In the definition of Δ-complex, we say that the restriction of any φ_j^k to each $\partial_i \Delta_k$ is a φ_l^{k-1}, but we have been ambiguous as to the underlying identification $\Delta_{k-1} \simeq \partial_i \Delta_k$.

We adopt the convention that if this identification respects the orientation, $(\varphi_j^k)_{|\partial_i \Delta_k}$ is represented by $\sigma_l^{k-1} \in \mathcal{C}_{k-1}^{\Delta}(X)$ and by $-\sigma_l^{k-1}$ otherwise. In what follows, we use the self-explanatory notation

- $[\varphi_j^k] = \sigma_j^k \in \mathcal{C}_k^\Delta(X)$;
- $[(\varphi_j^k)_{|\partial_i \Delta_k}] = \pm \sigma_l^{k-1} \in \mathcal{C}_{k-1}^\Delta(X)$ (depending on the orientation).

Definition 12 (Boundary operator)**.** *For all $k \leq n$, the boundary operator is the group homomorphism*

$$\partial_k^\Delta : \mathcal{C}_k^\Delta(X) \longrightarrow \mathcal{C}_{k-1}^\Delta(X)$$

defined for all $j \leq m_k$

$$\partial_k^\Delta\big([\varphi_j^k]\big) = \sum_{i=0}^{n} (-1)^{i+1} [(\varphi_j^k)_{|\partial_i \Delta_k}]$$

and extended by linearity to the whole of $\mathcal{C}_k^\Delta(X)$.

2.1.4 A bit of algebra

Definition 13 (Chain complex)**.** *A **chain complex** $\mathbf{C}$ is a sequence of abelian groups $(C_n)_{n\in\mathbb{N}}$ together with group homomorphisms*

$$\partial_n : C_n \longrightarrow C_{n-1}$$

such that $\forall n \in \mathbb{Z}$, $\partial_n \circ \partial_{n+1} = 0$.

The condition $\partial_n \circ \partial_{n+1} = 0$ is equivalent to the condition

$$\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n).$$

Definition 14 (Homology groups)**.** *The sequence of homology groups associated with the chain complex $\mathbf{C}$ are the groups, for $n \in \mathbb{N}$*

$$\mathrm{H}_n(\mathbf{C}) = \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1}).$$

These two fairly general definitions are the starting point of what is commonly referred to as *homological algebra*. This is a subject that is formally distinct from algebraic topology, although they are very much historically intertwine, as algebraic topology was one of the main motivations for its development.

An important point though is that there are important *homology theories* (that is a formal setting where homology groups can be defined) that are used in other mathematical areas.

2.1.5 Simplicial Homology

In this paragraph we show that for a Δ-complex X, the groups of $\mathcal{C}_k^\Delta(X)$ together with the boundary operators ∂_k^Δ form a chain complex.

Lemma 36. *For any Δ-complex X and for any $k \leq n$*

$$\partial_n \circ \partial_{n+1} = 0.$$

Proof. We just introduce some extra notation. For $j \neq l \leq n$, let $\partial_{jl}(\Delta_k) := \partial_j(\Delta_k) \cup \partial_l(\Delta_k)$ with the natural identification with Δ_{k-2}.

By definition,

$$\partial_k^\Delta\left([\varphi_j^k]\right) = \sum_{i=0}^{k} (-1)^{i+1} [(\varphi_j^k)_{|\partial_i \Delta_k}].$$

We compute $\partial_{k-1}^\Delta((\varphi_j^k)_{|\partial_i \Delta_k}])$. It is by definition

$$\partial_{k-1}^\Delta([(\varphi_j^k)_{|\partial_i \Delta_k}]) = \sum_{l<i} (-1)^{l+1} [(\varphi_j^k)_{|\partial_{il} \Delta_k}] + \sum_{l>i} (-1)^{l} [(\varphi_j^k)_{|\partial_{il} \Delta_k}].$$

We now get that

$$\partial_{k-1}^\Delta \circ \partial_k^\Delta\left([\varphi_j^k]\right) = \sum_{i=0}^{k} (-1)^{i+1} \sum_{l<i} (-1)^{l+1} [(\varphi_j^k)_{|\partial_{il} \Delta_k}] + \sum_{l>i} (-1)^{l} [(\varphi_j^k)_{|\partial_{il} \Delta_k}].$$

$$\partial_{k-1}^\Delta \circ \partial_k^\Delta\left([\varphi_j^k]\right) = \sum_{i \neq l}^{k} \epsilon(i,l) [(\varphi_j^k)_{|\partial_{il} \Delta_k}]$$

where $\epsilon(l,i) = (-1)^{l+i}$ if $l < i$ and $\epsilon(l,i) = (-1)^{l+i+1}$ if $l > i$. Grouping together terms for the form (i,l) and (l,i) we get

$$\partial_{k-1}^\Delta \circ \partial_k^\Delta\left([\varphi_j^k]\right) = \sum_{i<l}^{k} (\epsilon(i,l) + \epsilon(l,i) [(\varphi_j^k)_{|\partial_{il} \Delta_k}].$$

Since for all (i,l), $\epsilon(i,l) + \epsilon(l,i) = 0$ we get

$$\partial_{k-1}^\Delta \circ \partial_k^\Delta\left([\varphi_j^k]\right) = 0.$$

By linearity, $\partial_{k-1} \circ \partial_k \equiv 0$.

□

This shows that $(\mathcal{C}_k^\Delta, \partial_k)_{k \in \mathbb{N}}$ is a chain complex. We can therefore define its homology groups.

Definition 15 (Simplicial homology). *The **simplicial homology groups** of a Δ-complex X are the groups*

$$\mathrm{H}_n^\Delta(X) = \mathrm{Ker}(\partial_n^\Delta)/\mathrm{Im}(\partial_{n+1}^\Delta).$$

Dependence of the homology groups of the Δ-complex structure. We have here defined *homology groups* for a space X that depends on the way it is represented as a Δ-complex. A topological space X can be given different Δ-complex structures which yield *a priori* different *homology groups*. An important theorem that we will state (and maybe prove) shows that actually, the simplicial homology groups of a given space X **do not** depend on the Δ-complex structure chosen to compute them, and therefore are invariants of the topological space itself.

Exercise 18. *Show that for any Δ-complex X, $\mathrm{H}_0(X)$ is equal to the number of connected components of X.*

Exercise 19. *Put a Δ-complex structure on S^1 and compute its homology groups.*

Exercise 20. *Put a Δ-complex structure on S^n and compute its homology groups.*

Exercise 21. *Put a Δ-complex structure on $\mathbb{T}^n$ and compute its homology groups.*

Exercise 22. *Put a Δ-complex structure on $\mathbb{RP}^n$ and compute its homology groups.*

Exercise 23. *Put a Δ-complex structure on the Klein bottle and compute its homology groups.*

Exercise 24. *Put a Δ-complex structure on the Moebius strip and compute its homology groups.*

Exercise* 25. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$ and let $T_A : \mathbb{T}^2 \longrightarrow \mathbb{T}^2$ be the map it induces on the two-torus. Let $X := \mathbb{T}^2 \times_{T_A} S_1$ be the suspension of T_A. Put a Δ-complex structure on X and compute its homology groups.*

2.2 Singular homology

This paragraph covers the material discussed in Lecture 14.

In this paragraph we define the *singular homology groups*, which have the advantage of not depending of a Δ-structure on X (and which can be defined for an arbitrary topological space X). The main drawback is that they are hard to compute. A general theorem will ensure that the simplicial and singular homology groups are isomorphic. We will then rely on singular homology for abstract considerations and simplicial homology for calculation.

2.2.1 Definition

Singular chains Let X be a topological space. We define the set of singular chains $\mathcal{C}_n(X)$ to be the free abelian group generated by *continuous* maps $\varphi : \Delta_n \longrightarrow X$, *i.e.*

$$\mathcal{C}_n(X) := \bigoplus_{\varphi:\Delta_n \to X \text{ continuous}} \mathbb{Z} \cdot [\varphi].$$

Boundary operator We can therefore define a boundary operator in an analogous way to what we did for the simplicial homology. If $\varphi : \Delta_n \longrightarrow X$, we define $\partial_n[\varphi] \in \mathcal{C}_{n-1}(X)$ by

$$\partial_n[\varphi] := \sum_{i=0}^{n} (-1)^i \cdot [\varphi_{|\partial_i \Delta_n}]$$

where we use the standard identification between $\partial_i \Delta_n$ and Δ_{n-1} in order to view $\varphi_{|\partial_i \Delta_n}$ as a map $\Delta_{n-1} \to X$. We extend ∂_n linearly to defined

$$\partial_n : \mathcal{C}_n(X) \longrightarrow \mathcal{C}_{n-1}(X).$$

Proposition 37. $(\mathcal{C}_n(X), \partial_n)_{n\in\mathbb{N}}$ *is a chain complex, i.e. for all* $n \geq 2$, $\partial_{n-1} \circ \partial_n \equiv 0$.

Proof. Similar to the proof of Lemma 36. □

Definition 16 (Singular homology)**.** *The* ***singular homology*** *groups are the homology groups of the chain complex* $(\mathcal{C}_n(X), \partial_n)_{n\in\mathbb{N}}$ *which we denote by*

$$\mathrm{H}_n(X).$$

2.2.2 Comparison between Singular and Simplicial Homologies

In this paragraph we assume that X is a Δ-complex, that is comes with maps $(\varphi^k)_{i\in I_k} : \Delta_k \longrightarrow X$ for all $k \leq n$ which define its Δ-complex structure. We have a natural group homomorphism

$$i_k : \mathcal{C}_k^{\Delta}(X) \longrightarrow \mathcal{C}_k(X)$$

which maps $[\sigma_i^k]$ (the chain in $\mathcal{C}_k^{\Delta}(X)$ representing φ_i^k) to the chain in $\mathcal{C}_k(X)$ representing the continuous function $\varphi_i^k : \Delta_k \longrightarrow X$.

It ensues from the definition of i_k that it satisfies the following for all $k \geq 1$

$$i_{k-1} \circ \partial_k^{\Delta} = \partial_k \circ i_k.$$

It implies that i_k sends $\mathrm{Ker}(\partial_k^{\Delta})$ into $\mathrm{Ker}(\partial_k)$ and $\mathrm{Im}(\partial_{k+1}^{\Delta})$ into $\mathrm{Im}(\partial_{k+1})$. The maps i_k thus induce for all $k \in \mathbb{N}$

$$\varphi_k : \mathrm{H}_k^{\Delta}(X) \longrightarrow \mathrm{H}_k(X).$$

We will dedicate the next paragraphs to proving the fundamental theorem.

Theorem 38. *Let* X *be a finite* Δ*-complex. For all* $k \in \mathbb{N}$ *the map*

$$\varphi_k : \mathrm{H}_k^{\Delta}(X) \longrightarrow \mathrm{H}_k(X)$$

defined above is a group isomorphism.

2.2.3 Induced maps

In this paragraph we derive one of the nice consequence of the flexible nature of the singular homology.

Let X and Y be two topological spaces and let $f : X \longrightarrow Y$ be a continuous map. For any continuous maps $\varphi : \Delta_k \longrightarrow X$, $f \circ \varphi$ is a continuous map from Δ_k to Y. Extending linearly one sees that f induces a group homomorphism

$$f_* : \mathcal{C}_k(X) \longrightarrow \mathcal{C}_k(Y).$$

Moreover, since the boundary operator is defined by considering restrictions of maps φ to subsets of Δ_k we readily have

$$\partial_k^Y \circ f_* = f_* \circ \partial_k^X$$

where ∂_k^X and ∂_k^Y are the boundary operator acting on $\mathcal{C}_k(X)$ and $\mathcal{C}_k(Y)$ respectively. This implies that f_* induces

$$f_* : \mathrm{H}_k(X) \longrightarrow \mathrm{H}_k(Y)$$

for all $k \in \mathbb{N}$.

2.2.4 Relative homology

We introduce a concept that is instrumental to the proof of Theorem 38 but which is also of independent interest.
Let $A \subset X$ be a subset of X. We can define, for all $k \in \mathbb{N}$,

$$\mathcal{C}_k(A) \subset \mathcal{C}_k(X)$$

which is the subgroup of singular chains generated by continuous $\varphi : \Delta_k \longrightarrow A$ taking their values in A. We have the following obvious but important property

$$\forall k,\ \partial_k(\mathcal{C}_k(A)) \subset \mathcal{C}_{k-1}(A).$$

This implies that ∂_k induces

$$\partial_k : \mathcal{C}_k(X)/\mathcal{C}_k(A) \longrightarrow \mathcal{C}_{k-1}(X)/\mathcal{C}_{k-1}(A).$$

We use henceforth the notation

$$\mathcal{C}_k(X, A) := \mathcal{C}_k(X)/\mathcal{C}_k(A).$$

Since $\partial_k \circ \partial_{k+1} : \mathcal{C}_{k+1}(X) \to \mathcal{C}_{k-1}(X)$ is zero for all $k \geq 1$, it is also the case for the induced maps $\partial_k \circ \partial_{k+1} : \mathcal{C}_{k+1}(X, A) \to \mathcal{C}_{k-1}(X), A$. We thus obtain the following Proposition-Definition.

Proposition 39 (Relative homology groups)**.** *$(\mathcal{C}_k(X, A), \partial_k)_{k \in \mathbb{N}}$ is a chain complex. The **relative** homology groups*

$$\mathrm{H}_k(X, A)$$

are the homology groups of this chain complex.

2.3 Long exact sequence for relative homology groups

This paragraph covers the material discussed in Lecture 15.

It is natural to wonder how the relative homology groups $\mathrm{H}_k(X, A)$ relate to $\mathrm{H}_k(X)$ and $\mathrm{H}_k(A)$. The inclusion $i : A \longrightarrow X$ gives a natural way to map $\mathrm{H}_k(A)$ into $\mathrm{H}_k(X)$ and it would be nice if $\mathrm{H}_k(X, A)$ were equal to $\mathrm{H}_k(X)/\mathrm{H}_k(A)$. The situation is a bit more complicated than that (and in a sense it is reassuring, for otherwise relative homology groups would not contain much new information). We explain hereafter how these three sequences of homology groups $\left(\left(\mathrm{H}_k(X)\right)_k, \left(\mathrm{H}_k(A)\right)_k \text{ and } \left(\mathrm{H}_k(X, A)\right)_k\right)$ are related.

The inclusion maps The inclusion $i : A \longrightarrow X$ induces natural maps

$$i_* : \mathrm{H}_k(A) \longrightarrow \mathrm{H}_k(X).$$

The projection maps There is a natural projection, for all k,

$$\pi : \mathcal{C}_k(X) \longrightarrow \mathcal{C}_k(X, A) = \mathcal{C}_k(X)/\mathcal{C}_k(A).$$

By definition of the boundary operator for both chain complexes $(\mathcal{C}_k(X, A), \partial_k)$ and $(\mathcal{C}_k(X, A), \partial_k)$ we have

$$\partial \circ \pi = \pi \circ \partial.$$

We make several abuses of notation, using the symbol ∂ for the boundary operator of all chain complexes we are working with, as well as dropping the index k when there is no ambiguity. That being said, the relation $\partial \circ \pi = \pi \circ \partial$ implies that π induces a map

$$\pi : \mathrm{H}_k(X) \longrightarrow \mathrm{H}_k(X, A).$$

The boundary maps We now construct maps $\mathrm{H}_k(X, A) \longrightarrow \mathrm{H}_{k-1}(A)$ which are a bit less obvious. Consider an element $u \in \mathrm{H}_k(X, A)$. It is represented by a chain $c \in \mathcal{C}_k(X)$ such that $\partial c \in \mathcal{C}_{k-1}(A)$. We set

$$\partial u := [\partial_k c]$$

where $[\partial_k c]$ is the class of $\partial_k c$ in $\mathrm{H}_{k-1}(A)$ for any representative c of u. We want to show that ∂ induces a map (a group homomorphism actually) $\mathrm{H}_k(X, A) \longrightarrow \mathrm{H}_{k-1}(A)$. We therefore have three things to check to ensure that ∂ is well-defined :

- that $\partial_k c$ is actually a cycle;
- that $[\partial c]$ does not depend on the choice of a representative c of u.

The first point is immediate, $\partial_{k_1}(\partial_k c) = 0$ as we have already proven that $\partial_{k-1} \circ \partial_k \equiv 0$. Any other representative of u in $\mathcal{C}_k(X)$ can be written $c' = c + a + \partial_{k+1} b$ for some $b \in \mathcal{C}_{k+1}(X)$ and $a \in \mathcal{C}_k(A)$. Therefore

$$\partial_k(c') = \partial_k(c) + \partial_k(a) + \partial_k \circ \partial_{k+1}(b).$$

Since $\partial_k \circ \partial_{k+1} = 0$, $\partial_k(c') - \partial_k(c) = \partial_k(a)$ and since $\partial_k(a) \in \mathcal{C}_{k-1}(A)$, $[\partial_k(c') - \partial_k(c)] = 0 \in \mathcal{C}_{k-1}(X, A)$ which is what we wanted to prove. We have thus defined

$$\partial : \mathrm{H}_k(X, A) \longrightarrow \mathrm{H}_{k-1}(A).$$

The long exact sequence. Putting all these maps together, we obtain the following sequence

$$\cdots \xrightarrow{\partial} \mathrm{H}_k(A) \xrightarrow{i_*} \mathrm{H}_k(X) \xrightarrow{\pi} \mathrm{H}_k(X,A) \xrightarrow{\partial} \mathrm{H}_{k-1}(A) \xrightarrow{i_*} \cdots$$

Proposition 40. *The sequence*

$$\cdots \xrightarrow{\partial} \mathrm{H}_k(A) \xrightarrow{i_*} \mathrm{H}_k(X) \xrightarrow{\pi} \mathrm{H}_k(X,A) \xrightarrow{\partial} \mathrm{H}_{k-1}(A) \xrightarrow{i_*} \cdots$$

is a long exact sequence. Precisely, it means that wherever it makes sense

- $\mathrm{Im}(\partial) = \mathrm{Ker}(i_*)$*;*
- $\mathrm{Im}(i_*) = \mathrm{Ker}(\pi)$*;*
- $\mathrm{Im}(\pi) = \mathrm{Ker}(\partial)$*.*

Proof. $\mathrm{Im}(\partial) = \mathrm{Ker}(i_*)$ Let $a \in \mathrm{Im}(\partial)$. Then there exists a chain in $x \in \mathcal{C}_{k+1}(X)$ such that $a = [\partial_{k+1}x]$. Seen as a class in $\mathrm{H}_k(X)$, $a \in \mathrm{Ker}(\partial_{k+1})$, and therefore represents 0 by definition of $\mathrm{H}_k(X)$. Conversely, is a class $[a]$ represented by a chain $a \in \mathcal{C}_k(A)$ is in the kernel of i_*, it means that there exists $x \in \mathcal{C}_{k+1}(X)$ such that $a = \partial_{k+1}x$ and therefore $[a]$ is in the image of ∂.

$\mathrm{Im}(i_*) = \mathrm{Ker}(\pi)$ A class $[c]$ in $\mathrm{H}_k(X)$ is in the kernel of π if and only if there any representative $c \in \mathcal{C}_k(X)$ can be written $c = \partial_{k+1}c' + a$ with $c' \in \mathcal{C}_{k+1}(X)$ and $a \in \mathcal{C}_k(A)$. Since $\partial_k c = 0$, $\partial_k(a) = 0$ and therefore $[c]$ is in $\mathrm{Ker}(\pi)$ if and only if $[c]$ is in $\mathrm{Im}(i_*)$.

$\mathrm{Im}(\pi) = \mathrm{Ker}(\partial)$ A class $[c]$ in $\mathrm{H}_k(X,A)$ if and only if there exists $c \in \mathcal{C}_k(X)$ representing c such that $\partial_k c = 0$. This is equivalent to $\partial([c]) = 0$.

□

2.4 Barycentric subdivision

This paragraph covers the material discussed in Lecture 16.

Let X be a topological space, and let $\mathcal{U} = \{U_i\}_{i\in I}$ a collection of open sets such that $\bigcup_i U_i = X$. We define

$$\mathcal{C}_n^{\mathcal{U}}(X)$$

the subgroup of $\mathcal{C}_n(X)$ of generated by singular chains $\varphi : \Delta \longrightarrow U_i$ taking values in one of the U_i. For all n,

$$\partial_n(\mathcal{C}_n^{\mathcal{U}}(X)) \subset \mathcal{C}_{n-1}^{\mathcal{U}}(X)$$

which implies that

$$(\mathcal{C}_n^{\mathcal{U}}(X), \partial_n)_{n\geq 0}$$

is a chain complex. We denote by $\mathrm{H}_n^{\mathcal{U}}(X)$ its homology groups. In this section we prove the following important technical theorem.

Theorem 41. *The inclusion $i : \mathcal{C}_n^{\mathcal{U}}(X) \longrightarrow \mathcal{C}_n(X))$ induces an isomorphism*

$$\mathrm{H}_n^{\mathcal{U}}(X) \longrightarrow \mathrm{H}_n(X)$$

for all $n \geq 0$.

2.4.1 Subdivison of the simplex Δ_n

Recall that the definition of Δ_n

$$\Delta_n := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_i x_i = 1 \text{ and } \forall i, x_i \geq 0\}.$$

The barycentre of Δ_n is the point $b_n = (\frac{1}{n+1}, \frac{1}{n+1}, \cdots, \frac{1}{n+1})$. We define a subdivision of Δ_n inductively the following way.

- $n = 1$, b is the middle point of a segment homeomorphic to $[0, 1]$. The subdivision of $[0, 1]$ we are considering is

  $$[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1].$$

  Note that in particular this subdivision gives Δ_1 a particular Δ complex structure.

- Assume that we have carried out the procedure for Δ_n, meaning that we have written $\Delta_n = \bigcup_{i=1}^{(n+1)!} \Delta_n^i$, where the inclusions $\Delta_n^i \longrightarrow \Delta_n$ give Δ_n a structure of Δ-complex. The facets of Δ_{n+1} are $n + 2$ copies of Δ_n. For every simplex σ of the subdivison of Δ_n, consider the simplex spanned by σ and b_{n+1}. We thus obtain $(n + 2) \cdot (n + 1)!$ simplices which cover Δ_n and only intersect along entire subsimplices.

(It is highly recommended to draw a picture of what happens when $n = 2$ to get a clear mental image of this procedure). We identify linearly each Δ_n^i with Δ_n in such a way that the identification preserves the orientation.

Definition 17 (Barycentric subdivision)**.** *The partition $\Delta_n = \bigcup_{i=0}^{n} \Delta_n^i$ is called the **barycentric subdivision** of Δ_n. By extension, the image of this partition for any Δ'_n image of Δ_n by a **linear** map is also called the barycentric subdivision of Δ'_n.*

2.4.2 Subdivision of singular chains

Let $\varphi : \Delta_n \longrightarrow X$ be a continuous function. Thinking of φ as a singular chain, we define

$$S([\varphi]) = \sum_{i=1}^{(n+1)!} [\varphi_{|\Delta_n^i}].$$

Proposition 42. *For all n,*

$$\partial_n \circ S = S \circ \partial_n.$$

Proof. The proof goes by induction on n.

For $\Delta_1 = [0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$ and $\varphi : \Delta_1 \longrightarrow X$ we have that $\partial\varphi$ is the restriction of φ to $\{1\}$ minus the restriction of φ to $\{0\}$, whereas $\partial S(\varphi)$ is $\varphi_{|\{1\}} - \varphi_{|\{\frac{1}{2}\}} + \varphi_{|\{\frac{1}{2}\}} - \varphi_{|\{0\}}$

Assume that the Proposition is proved for $k \leq n$. Let $\varphi : \Delta_{n+1} \longrightarrow X$ We group the boundary terms of $\partial S(\varphi)$ depending on whether they are the restriction of φ to a facet of Δ^i_{n+1} that belongs to $\partial\Delta_{n+1}$ or which intersects the interior of Δ_{n+1}. We can thus write

$$\partial S(\varphi) = \sum_{j=0}^{n+2} S(\varphi_{\partial_j \Delta_{n+1}}) + A$$

where A is a sum of terms of the form $\varphi_{\partial_l \Delta^i_{n+1}}$ where the face $\partial_l \Delta^i_{n+1}$ intersects the interior of Δ_{n+1}. Any such face is the intersection of exactly two subsimplices Δ^i_{n+1} and $\Delta^{i'}_{n+1}$ with opposite induces orientations. Thus A is a sum of terms which can be re-arranged to have pairwise cancellation. Finally, $\sum_{j=0}^{n+2} S(\varphi_{\partial_j \Delta_{n+1}}) = S(\partial\varphi)$ which proves

$$\partial \circ S = S \circ \varphi.$$

□

S extends linearly to $\mathcal{C}_n(X)$ for all n to define a group homomorphism

$$S : \mathcal{C}_n(X) \longrightarrow \mathcal{C}_n(X).$$

The intuitive meaning of S is the following : we take a chain represented by a certain number of singular simplices which we all replace by a finite number of *smaller* simplices. The goal is to work with an arbitrary chain c, and apply S sufficiently many times so that all the simplices appearing in the decomposition of $S^l(c)$ take their values in an open set of the covering $\mathcal{U}$.

The important Proposition is the following,

Proposition 43. *For any $\varphi : \Delta_n \longrightarrow X$, there exists a chain $\sigma \in \mathcal{C}_{n+1}(X)$ such that*

$$\partial(c) = [\varphi] - \sum_{i=1}^{(n+1)!} [\varphi_{|\Delta^i_n}] = [\varphi] - S([\varphi]).$$

Proof. We only give the proof in the case $n = 1$ to spare ourselves some technicalities. The full detail of the proof can be found in Hatcher. Identifies Δ_2 with an isosceles triangle, and consider π the orthogonal projection on the base restricted to Δ_2. The base naturally identifies with $[0,1]$. The other two sides of Δ_2 are mapped onto $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ respectively.

Now consider any continuous $\varphi : \Delta_1 = [0,1] \longrightarrow X$. One can easily check that $\partial(\varphi \circ \pi)$ is a formal sum of restriction of φ to $[0,1]$, $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$ respectively. Taking orientation into account we find

$$\partial(\varphi \circ \pi) = \varphi - \varphi_{|[0,\frac{1}{2}]} - \varphi_{|[\frac{1}{2},1]} = \varphi - S(\varphi).$$

An elaboration of this argument gives the result for all n. □

Proposition 44. *For any continuous $\varphi : \Delta_n \longrightarrow X$, there exists $l \geq 0$ such that*

$$S^l([\varphi]) \in \mathcal{C}_n^{\mathcal{U}}(X).$$

We will build upon the following Lemma.

Lemma 45. *For any continuous $\varphi : \Delta_n \longrightarrow X$, there exists ϵ such that for any $x \in \Delta_n$, there exists i such that*

$$\varphi(\mathrm{B}(x, \epsilon)) \subset U_i.$$

Proof. Assume it is not the case and that for any k there exists $x_k \in \Delta_n$ such that $\varphi(\mathrm{B}(x_k, \frac{1}{k}))$ is contained in no U_i. By compactness of Δ_k, we can (up to extracting a subsequence) assume that $x_k \to x_\infty \in \Delta_n$. Let i be such that $\varphi(x_\infty) \in U_i$. By continuity of φ, $\varphi^{-1}(U_i)$ contains the ball centred at x_∞ of radius ϵ_0 for some ϵ_0. For k sufficiently large, $\mathrm{B}(x_k, \frac{1}{k}) \subset \mathrm{B}(x_\infty, \epsilon_0)$, which contradicts the fact that $\varphi(\mathrm{B}(x_k, \frac{1}{k}))$ is contained in no U_i. □

Proof of Proposition 44. By definition $S^l(\varphi)$ is a linear combination of restrictions of φ to a partition of Δ_n obtained by subdividing Δ_n l times. We check that for any Δ_n^i element of the barycentric division of a simplex Δ_n,

$$\mathrm{diam}(\Delta_n^i) \leq \frac{n}{n+1} \mathrm{diam}(\Delta_n).$$

We proceed by induction on n. For $n = 1$, the inequality is straightforward. Assume that the inequality has been established for $n - 1$. The diameter of a simplex is the supremum of the distance between to simplices. Considering two vertices of Δ_n^i, either they are vertices of a copy of Δ_{n-1}^j in which case we can apply the induction hypothesis. Otherwise one of these vertices is b_n and the other is on the facets of Δ_n^i which is isomorphic to Δ_{n-1}^j. Let a be such a vertex. Without loss of generality we can assume that $a = (1, 0, 0 \cdots, 0)$.

$$|a - b_n| = |(\frac{1}{n+1}, \cdots, \frac{1}{n+1}) - (1, 0, 0 \cdots, 0)| = \frac{n}{n+1} |((1, 0, 0 \cdots, 0) - (0, \frac{1}{n}, \cdots, \frac{1}{n})|$$

$$|a - b_n| \leq \frac{n}{n+1} \mathrm{diam}(\Delta_n).$$

Consider l such that $(\frac{n}{n+1})^l \leq \epsilon$ from Lemma 45. $S^l([\varphi])$ is thus a linear combination of restriction of φ to sub-simplices each taking values in one of the U_is, which completes the proof of the Proposition.

□

Proof of Theorem 41 Building on Proposition 42, Proposition 43 and Proposition 44, we prove that the natural map

$$\mathrm{H}_n^{\mathcal{U}}(X) \longrightarrow \mathrm{H}_n(X)$$

is a group isomorphism.

Injectivity. Let $[c]$ be a class in $\mathrm{H}_n^{\mathcal{U}}(X)$, where $c \in \mathcal{C}_n^{\mathcal{U}}(X)$ is such that $\partial_n c = 0$. Assume that the class defined by c is trivial in $\mathrm{H}_n(X)$. This means that there exists $u \in \mathcal{C}_{n+1}(X)$ such that

$$c = \partial_{n+1}(u).$$

We show by induction on l that for all $l \geq 0$, $c = \partial(S^l(u))$. It is clearly true for $l = 0$. Assume that the result is true for l. By Proposition 43, there exists $a \in \mathcal{C}_{n+2}(X)$ such that $S^{l+1}(u) = S^l(u) + \partial_{n+2}a$. Thus

$$\partial_{n+1}(S^{l+1}(u)) = \partial_{n+1}(S^l(u)) + \partial_{n+1} \circ \partial_{n+2}(a) = \partial u$$

which proves the partial result. But by Proposition 44, there exists $l > 0$ such that $S^l(u) \in \mathcal{C}_{n+1}^{\mathcal{U}}(X)$; since $c = \partial(S^l(u))$, $[c]$ is trivial in $\mathrm{H}_n^{\mathcal{U}}(X)$. This implies that $\mathrm{H}_n^{\mathcal{U}}(X) \longrightarrow \mathrm{H}_n(X)$ as we have just proven that its kernel is trivial.

Surjectivity. We want to show that any class $[c]$ in $\mathrm{H}_n(X)$ can be represented by a chain in $\mathcal{C}_n^{\mathcal{U}}(X)$. Take an arbitrary representative $c \in \mathcal{C}_n(X)$. We show by induction that $[S^l(c)]$ represents $[c]$. Clearly true for $l = 0$. If $[S^l(c)] = [c]$ we use the fact that there exists $a \in \mathcal{C}_{n+1}(X)$ such that $S^{l+1}(c) = S^l(c) + \partial_{n+1}a$ (by Proposition 43) which implies that $[S^{l+1}(c)] = [S^l(c)] = [c]$ in $\mathrm{H}_n(X)$. Since there exists $l > 0$ such that $S^l(c) \in \mathcal{C}_n^{\mathcal{U}}(X)$, $[c]$ can be represented by a cycle in $\mathcal{C}_n^{\mathcal{U}}(X)$ which proves surjectivity.

2.5 Excision

The next two paragraphs cover the material discussed in Lecture 17. In this paragraph we apply some of the methods developed in the previous section to prove the following Theorem.

Theorem 46 (Excision theorem)**.** *Let X a topological space, $A \subset X$ and Z a subset of X contained in the interior of A. Then the inclusion $i : (X \setminus Z, A \setminus Z) \longrightarrow (X, A)$ induces an isomorphism between relative homology groups*

$$i_* : \mathrm{H}_n(X \setminus Z, A \setminus Z) \longrightarrow \mathrm{H}_n(X, A)$$

for all $n \geq 0$.

Proof. We set $B = X \setminus Z$. In particular

- $A \setminus Z = A \cap B$;
- $\mathrm{Int}(A) \cup \mathrm{Int}(B) = X$ (as $Z \subset \mathrm{Int}(A)$, and $\overline{Z} = X \setminus \mathrm{Int}(B)$).

Therefore the Excision theorem can be reformulated

$$i_* : \mathrm{H}_n(B, A \cap B) \longrightarrow \mathrm{H}_n(X, A)$$

is an isomorphism for all n. Consider the cover of X by open sets $\mathrm{Int}(\mathcal{A}), \mathrm{Int}(\mathcal{B})$.

Injectivity Let $[c]$ be a class in $\mathrm{H}_n(B, A \cap B)$, where $c \in \mathcal{C}_n(B)$ for which $\partial_n c \in \mathcal{C}_{n-1}(A \cap B)$. Assume that c defines a class $[c]$ that is trivial in $\mathrm{H}_n(X, A)$, which would mean that there exists $d \in \mathcal{C}_{n+1}(X)$ such that $c - \partial d \in \mathcal{C}_n(A)$.

Using Propositions 43 and 44 we can show that there exists $l > 0$ such that

$$c - \partial S^l(d) \in \mathcal{C}_n(A)$$

and $S^l(d) \in \mathcal{C}^U_{n+1}(X)$. $S^l(d)$ can thus be written $a + b$ where $a \in \mathcal{C}_{n+1}(A)$ and $b \in \mathcal{C}_{n+1}(B)$. This implies that $c - \partial_n(b) \in \mathcal{C}_n(A)$ but by definition is also in $\mathcal{C}_n(B)$ which implies that it is in $\mathcal{C}_n(A \cap B)$. This implies that $[c]$ is trivial in $\mathrm{H}_n(B, A \cap B)$.

Surjectivity Let $[c]$ be in $\mathrm{H}_n(X, A)$ represented by $c \in \mathcal{C}_n(X)$ such that $\partial_n c \in \mathcal{C}_{n-1}(A)$. Again, using Propositions 43 and 44 we can subdivide c as to write it $c = S^l(c) + d$ with $d = \partial_{n+1}(e)$ for some $e \in \mathcal{C}_{n+1}(X)$ and $S^l(d) \in \mathcal{C}^U_n(X)$. We write $S^l(d) = a + b$ where $a \in \mathcal{C}_n(A)$ and $b \in \mathcal{C}_n(B)$.

Now

$$\partial_n b = \partial_n c - \partial_n a - \partial_n \circ \partial_{n+1}(e) = \partial_n c - \partial_n a.$$

This shows that $\partial_n b \in \mathcal{C}_{n-1}(A)$ and is by definition in $\mathcal{C}_{n-1}(B)$. This shows that

- b defines a class in $\mathrm{H}_n(B, A \cap B)$;
- $\partial(c - b) \in \mathcal{C}_{n-1}(A)$ and therefore c and b represent the same class in $\mathrm{H}_n(X, A)$.

This completes the proof of surjectivity and thus that of the Excision theorem.

□

2.6 Singular homology and homotopy

In this paragraph we develop some more tools to compute the singular homology of a space.

Let X and Y two topological spaces. Two continuous maps $f, g : X \longrightarrow Y$ are said to be *homotopic* if there exists a continuous

$$H : [0,1] \times X \longrightarrow Y$$

such that

- $H(0, \cdot) = f$;
- $H(0, \cdot) = g$.

The main purpose of this paragraph is to prove the following Proposition and derive from it a few important corollaries.

Proposition 47. *Let f and g be two homotopic continuous maps $X \longrightarrow Y$. Then for any $n \in \mathbb{N}$, the induced maps*

$$f_*, g_* : \mathrm{H}_n(X) \longrightarrow \mathrm{H}_n(Y)$$

are equal.

The proof goes along the following line. For any cycle $c \in \mathcal{C}_n(X)$, we associate a chain u in $\mathcal{C}_{n+1}(X)$ which is to be thought of as $c \times [0,1]$. We post compose u by H the homotopy from f to g to obtain u' such that

$$\partial_{n+1} u' = f \circ c - g \circ c$$

which will prove that $f \circ c$ and $g \circ c$ represent the same class in $\mathrm{H}_n(Y)$.

Subdivision of $\Delta_n \times [0,1]$ We first start by giving a Δ-complex structure to $\Delta_n \times [0,1]$. Seeing Δ_n as a convex subset of $\mathbb{R}^n$, we embed $\Delta_n \times [0,1]$ in $\mathbb{R}^{n+1}$. We thus see $\Delta_n \times [0,1]$ as a convex polytopes with vertices $e_0, \cdots, e_n, f_0, \cdots, f_n$ where $e_0, \cdots, e_n$ are the vertices of $\Delta_n \times \{0\}$ and $f_0, \cdots, f_n$ are those of $\Delta_n \times \{1\}$.

We define $\Delta_{n+1}(i)$ to be the convex hull/the simplex spanned by $e_0, e_1, \cdots, e_i, f_i, \cdots, f_n$. The following things are easily checked :

- $\Delta_{n+1}(i)$ is the image by a linear map of the standard simplex Δ_n;
- two consecutive $\Delta_{n+1}(i)$s intersect along one of their faces, other wise they do intersect along lower subsimplices;
- the faces of $\Delta_{n+1}(i)$ which are not faces of $\Delta_{n+1}(i-1)$ or $\Delta_{n+1}(i+1)$ belong to the boundary of $\Delta_n \times [0,1]$.

We will denote by $[a_1, \cdots, a_k]$ the convex hull of the k-points a_is, (so that $\Delta_{n+1}(i) = [e_0, e_1, \cdots, e_i, f_i, \cdots, f_n]$) together with the orientation induced by the ordering of the a_is.

A chain bounding f_* **and** g_* Let $\alpha : \Delta_n \longrightarrow X$ be a chain in $\mathcal{C}_n(X)$, and let $H : [0,1] \times X \longrightarrow Y$ be a homotopy between f and g. Define

$$\begin{array}{rcccc} B(\alpha) & := & \Delta_n \times [0,1] & \longrightarrow & Y \\ & & (x,t) & \longmapsto & H(t, \alpha(x)) \end{array}.$$

By formally taking the sums of the restrictions of $B(\alpha)$ to all the $\Delta_{n+1}(i) \subset \Delta_n \times [0,1]$ we can see $B(\alpha)$ as an element of $\mathcal{C}_{n+1}(Y)$. We extend B linearly to all elements $\mathcal{C}_n(X)$ to obtain a group homomorphim

$$B_n : \mathcal{C}_n(X) \longrightarrow \mathcal{C}_{n+1}(Y).$$

Lemma 48. *For all* $n \in \mathbb{N}$,

$$\partial_{n+1} \circ B_n = g_* - f_* + B_{n-1} \circ \partial_n.$$

Proof. It suffices to check it for a singular chain $\alpha : \Delta_n \longrightarrow X$.

$$B_n(\alpha) := \sum_{i=0}^{n} (H(\alpha \times \mathrm{Id}))_{|\Delta_{n+1(i)}}.$$

This way we get

$$\partial_{n+1} B_n(\alpha) = \sum_{i=0}^{n} (\partial_{n+1} H(\alpha \times \mathrm{Id}))_{|\Delta_{n+1(i)}}.$$

Now,

$$\partial_{n+1}\Delta_{n+1}(i) = \sum_{j=0}^{i-1} (-1)^j [e_0, \cdots, e_{j-1}, e_{j+1}, \cdots, e_i, f_i, \cdots, f_n] + (-1)^i [e_0, e_{i-1}, f_i, \cdots, f_n]$$

$$+(-1)^{i+1}[e_0, e_i, f_{i+1}, \cdots, f_n] + \sum_{j=i+1}^{n} (-1)^{j+1} [e_0, \cdots, e_i, f_i, \cdots, f_{j-1}, f_{j+1}, \cdots, f_n]$$

When summing up all these formal combinations of simplices, the two central terms cancel out with central terms of $\partial_{n+1}\Delta_{n+1}(i-1)$ and $\partial_{n+1}\Delta_{n+1}(i+1)$ apart from $[e_0, \cdots, e_n] = \Delta_n \times \{0\}$ and $-[f_0, \cdots, f_n] = -\Delta_n \times \{1\}$. The other terms, in the sums, can be grouped together to recognised the subdivisions of $(\partial_i \Delta_n) \times [0,1]$. By taking restrictions of $H(\alpha \times \mathrm{Id})$ to these subsimplices, we get that

$$\partial_{n+1} B_n(\alpha) = f_*\alpha - g_*\alpha + B_{n-1}(\partial_n \alpha).$$

□

Proof of Proposition 47. Let α a representative of a class $[\alpha] \in \mathrm{H}_n(X)$. This means that $\partial_n \alpha = 0$. By the previous Lemma,

$$\partial_{n+1} B_n(\alpha) = f_*\alpha - g_*\alpha + B_{n-1}(\partial_n \alpha)$$

but since $\partial_n \alpha = 0$,

$$\partial_{n+1} B_n(\alpha) = f_*\alpha - g_*\alpha$$

which implies that

$$f_*[\alpha] = g_*[\alpha].$$

2.7 Homology of a quotient

This paragraph covers the material discussed in Lecture 18.

In this paragraph we consider a topological space X, a subset $A \subset X$, and try to relate the homology groups $\mathrm{H}_n(X)$, $\mathrm{H}_n(A)$ and $\mathrm{H}_n(X/A)$. We first introduce two important technical notions.

2.7.1 Quotient by a subspace

Let X be any topological space and A a subset of X. We introduce the following equivalence relation of X:

$$x \sim_A y \Leftarrow x = y \text{ or both } x \text{ and } y \in A.$$

Note that the natural projection (by post-composition of maps $\varphi : \Delta_n \to X$ by the projection $\pi : X \to X/A$) defines a group homomorphism $\mathcal{C}_n(X) \longrightarrow \mathcal{C}_n(X/A)$ which induces

$$\mathcal{C}_n(X, A) \longrightarrow \mathcal{C}_n(X/A, A/A)$$

which commutes with the boundary operators ∂_n. In particular it induces a natural quotient map

$$q : \mathrm{H}_n(X, A) \longrightarrow \mathrm{H}_n(X/A, A/A) \simeq \mathrm{H}_n(X/A).$$

Definition 18 (Quotient of X by A)**.** *The quotient of X by A, which we denote by X/A is the space $X/_{\sim_A}$ endowed with quotient topology.*

2.7.2 Deformation retractions

Let $A \subset X$.

Definition 19 (Deformation retraction)**.** *A **deformation retraction** of X onto A is a map*

$$D : [0, 1] \times X \longrightarrow A$$

such that

1. *$\forall t \in [0, 1]$, $D(t, \cdot)$ restricted to A is the identity;*
2. *$D(0, \cdot) = \mathrm{Id}_X$;*
3. *$D(1, X) \subset A$.*

If there exists a deformation retraction of X onto A, we say that X *deformation retracts* onto A. Note that a deformation retraction is a very particular case of homotopy.

Proposition 49. *Assume that X deformation retracts onto A. Then for all $n \in \mathbb{N}$ the inclusion $i : A \to X$ induces isomorphisms*

$$i_* : \mathrm{H}_n(A) \longrightarrow \mathrm{H}_n(X).$$

Proof. Let D be a deformation retraction of X onto A. By Proposition 47, $D(1, \cdot)$ induces the identity on $\mathrm{H}_n(X)$. Moreover, $D(1, \cdot)$ maps $\mathrm{H}_n(X)$ onto the image of $\mathrm{H}_n(A)$ by the inclusion of A in X since $D(1, \cdot)(X) \subset A$. Finally, the restriction of $D(1, \cdot)$ to A being the identity of A, $(D(1, \cdot))_*$ restricted to to image of $\mathrm{H}_n(A)$ in $\mathrm{H}_n(X)$ is the inverse of the application induced by the inclusion. □

2.7.3 Homology of a quotient versus relative homology

In this paragraph we give a proof of the following result.

Proposition 50. *Assume that A is a closed subset of X such that there exists $U \subset X$ an open neighbourhood of A such A deformation retracts onto A. The natural map defined above*

$$\mathrm{H}_n(X, A) \longrightarrow \mathrm{H}_n(X/A)$$

is an isomorphism for all $n \in \mathbb{N}$.

In order to prove this proposition, we will need to generalise the exact sequence in homology.

Exact sequence in relative homology. Let $B \subset A \subset X$ two subsets of a topological space X. The inclusions

$$i : (A, B) \longrightarrow (X, B)$$

and

$$i' : (X, B) \longrightarrow (X, A)$$

do induce a short exact sequence

$$0 \longrightarrow \mathcal{C}_n(A, B) \longrightarrow \mathcal{C}_n(X, B) \longrightarrow \mathcal{C}_n(X, A) \longrightarrow 0.$$

We leave it to the reader to check that the arguments of paragraph 2.3 generalise to prove the following

Proposition 51. *The sequence*

$$\cdots \xrightarrow{\partial} \mathrm{H}_k(A, B) \xrightarrow{i_*} \mathrm{H}_k(X, B) \xrightarrow{i'_*} \mathrm{H}_k(X, A) \xrightarrow{\partial} \mathrm{H}_{k-1}(A, B) \xrightarrow{i_*} \cdots$$

is a long exact sequence.

Proof of Proposition 50. We consider the spaces $A \subset U \subset X$ and their counterparts in the quotient $A/A \subset U/A \subset X/A$. The inclusions of respective pairs induce a certain number of natural maps in homology that we will need :

- $i_1 : \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, U)$;
- $i_2 : \mathrm{H}_n(X/A, A/A) \to \mathrm{H}_n(X/A, U/A)$;
- $j_1 : \mathrm{H}_n(X \setminus A, U \setminus A) \to \mathrm{H}_n(X, V)$;
- $j_2 : \mathrm{H}_n((X/A \setminus A/A), V/A \setminus A/A) \to \mathrm{H}_n(X/A, U/A)$;

as well as the maps induced by the quotient maps

- $q : \mathrm{H}_n(X, A) \to \mathrm{H}_n(X/, A/A)$;
- $q' : \mathrm{H}_n(X \setminus A, U \setminus A) \to \mathrm{H}_n(X/A \setminus A/A, U/A \setminus A/A)$.

By the excision theorem, j_1 and j_2 are both isomorphisms. U deformation retracts onto A (and U/A deformation retracts onto A/A) so by the exact sequence in relative homology

$$\mathrm{H}_n(U, A) = 0 \longrightarrow \mathrm{H}_n(X, A) \longrightarrow \mathrm{H}_n(X, U) \longrightarrow \mathrm{H}_{n-1}(U, A) = 0$$

(and the same for $X/A, U/A$ and A/A) we get that i_1 and i_2 are isomorphisms. Finally q' is an isomorphism since the quotient map $X \longrightarrow X/A$ is an homeomorphism from $X \setminus A$ onto $X/A \setminus A/A$. Finally, since all these maps are induced by inclusions and passing to the quotient which commute whenever they are defined, we have

$$q = i_2^{-1} \circ j_2 \circ q' \circ j_1^{-1} \circ i_1.$$

Thus q is an isomorphism.

2.8 Isomorphism between simplicial and singular homology

This paragraph covers the material discussed in Lecture 19. For the rest of this paragraph, X is a finite n-dimensional Δ complex. We establish the fundamental theorem

Theorem 52 (Equivalence of simplicial and singular homology groups)**.** *The natural maps between simplicial and singular homology groups*

$$\mathrm{H}_n^{\Delta}(X) \longrightarrow \mathrm{H}_n(X)$$

are isomorphisms for all $n \in \mathbb{N}$.

2.8.1 Some preliminary calculations

We apply some of the machinery that we have so far developed to compute (singular) homology groups of simple spaces; and from which we will derive Theorem 52.

Singular homology of points. If $X = \{p\}$ is a point, there is only one map $\varphi_: \Delta_k \longrightarrow X$ and thus $\mathcal{C}_k(X) = \mathbb{Z} \cdot [\varphi_k] \simeq \mathbb{Z}$ for all $k \geq 0$. $\partial_k[\varphi_k]$ is an alternate sum of φ_{k-1} with $k+1$ term, we thus get that

$$\partial_k[\varphi_k] = 0 \text{ if } k \text{ is odd}$$

and

$$\partial_k[\varphi_k] = [\varphi_{k-1}] \text{ if } k \text{ is even.}$$

At any rate, for all $k \geq 1$, $\mathrm{Im}(\partial_{k+1}) = \mathrm{Ker}(\partial_k)$ which gives

$$\mathrm{H}_k(X) = 0$$

for all $k \geq 1$ and $\mathrm{H}_0(X) = \mathbb{Z}$. Same reasoning yields $\mathrm{H}_k(X) = 0$ for all $k \geq 1$ and $\mathrm{H}_0(X) = \mathbb{Z}^l$ for X a set with l points endowed with the discrete topology. In particular

$$\mathrm{H}_k^{\Delta}(X) \longrightarrow \mathrm{H}_k(X)$$

is an isomorphism in this case.

Singular homology of the simplex. We derive from this discussion the calculation of homology groups of the simplex. Since Δ_k deformation retracts to a point for all $k \geq 0$, we have $\mathrm{H}_n(\Delta_k) = 0$ for $n \geq 1$ and $\mathrm{H}_0(\Delta_k) = \mathbb{Z}$. This again implies that

$$\mathrm{H}_k^{\Delta}(\Delta_k) \longrightarrow \mathrm{H}_k(\Delta_k)$$

is an isomorphism for the trivial Δ-complex structure on Δ_k.

2.8.2 Simplicial relative homology

Let X be a Δ-complex of dimension k. A subset of X is a Δ-subcomplex of dimension j if A is the union of images of maps defining the structure of X, $\varphi : \Delta_i \longrightarrow X$ with $i \leq j$. For any n, $\mathcal{C}_n^{\Delta}(X)$ contains the subgroup $\mathcal{C}_n^{\Delta}(A)$ generated by all Δ-simplices of dimension n taking their values in A. By definition $\partial_n(\mathcal{C}_n^{\Delta}(A)) \subset \mathcal{C}_{n-1}^{\Delta}(A)$. We can thus define the *simplicial relative homology groups* as the homology groups of the chain complex

$$\partial_n : \mathcal{C}_n^{\Delta}(X)/\mathcal{C}_n^{\Delta}(A) \longrightarrow \mathcal{C}_{n-1}^{\Delta}(X)/\mathcal{C}_{n-1}^{\Delta}(A).$$

We denote them by

$$\mathrm{H}_n^{\Delta}(X, A)$$

and the natural inclusion $\mathcal{C}_n^{\Delta}(X) \longrightarrow \mathcal{C}_n X)$ induces an inclusion

$$\mathrm{H}_n^{\Delta}(X, A) \longrightarrow \mathrm{H}_n(X, A).$$

2.8.3 Proof of Theorem 52

For any $k \leq n$, we denote by $X_k \subset X$ the union of all simplices of dimension k. We therefore have

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X.$$

In particular X_k is a finite l-dimensional Δ complex. We call X_k the *k-skeleton* of X. We prove the following statement by induction on k

$$\textbf{For all } n \in \mathbb{N}\textbf{,}\ \mathrm{H}_n^{\Delta}(X_k) = \mathrm{H}_n(X_k)$$

The statement holds for $k = 0$ by the calculation of the homology of points performed in the previous paragraph. We assume that the statement has been established for all $i \leq k - 1$. For all k, n, the inclusion $i : \mathcal{C}_n^{\Delta}(X_k) \longrightarrow \mathcal{C}_n(X_k)$ induces maps

$$i_* : \mathrm{H}_n^{\Delta}(X_k) \longrightarrow \mathrm{H}_n(X_k)$$

and

$$i_* : \mathrm{H}_n^{\Delta}(X_k, X_{k-1}) \longrightarrow \mathrm{H}_n(X_k, X_{k-1}).$$

These maps are such the following diagram commutes (where the top and bottom lines come from the long exact sequence in relative homology).

$$\begin{array}{ccccccccc}
\mathrm{H}_{n+1}^{\Delta}(X_k, X_{k-1}) & \longrightarrow & \mathrm{H}_n^{\Delta}(X_{k-1}) & \longrightarrow & \mathrm{H}_n^{\Delta}(X_k) & \longrightarrow & \mathrm{H}_n^{\Delta}(X_k, X_{k-1}) & \longrightarrow & \mathrm{H}_{n-1}^{\Delta}(X_{k-1}) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}_{n+1}(X_k, X_{k-1}) & \longrightarrow & \mathrm{H}_n(X_{k-1}) & \longrightarrow & \mathrm{H}_n(X_k) & \longrightarrow & \mathrm{H}_n(X_k, X_{k-1}) & \longrightarrow & \mathrm{H}_{n-1}(X_{k-1})
\end{array}$$

The heart of the argument is to show that the four downwards arrow i_*, $\mathrm{H}_{n+1}^{\Delta}(X_k, X_{k-1}) \to \mathrm{H}_{n+1}(X_k, X_{k-1})$, $\mathrm{H}_n^{\Delta}(X_{k-1}) \to \mathrm{H}_n(X_{k-1})$, $\mathrm{H}_n^{\Delta}(X_k, X_{k-1}) \to \mathrm{H}_n(X_k, X_{k-1})$ and $\mathrm{H}_{n-1}^{\Delta}(X_{k-1}) \to \mathrm{H}_{n-1}(X_{k-1})$ are isomorphism; and invoke a general algebra Lemma to conclude that the fifth downwards arrow

$$\mathrm{H}_n^{\Delta}(X_k) \to \mathrm{H}_n(X_k)$$

which is the one we care about, is an isomorphism.

Lemma 53 (The five Lemma)**.** *Assume we have the following commutating diagram between groups*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

and that

- *the top and bottom rows are exact sequences;*
- *the maps $A \to A'$, $B \to B'$, $D \to D'$ and $E \to E'$ are isomorphisms.*

Then the map $C \longrightarrow C'$ is an isomorphism.

We postpone the proof of the Lemma to next paragraph and carry on with our business. By the induction hypothesis, we know that

$$\mathrm{H}_n^{\Delta}(X_{k-1}) \longrightarrow \mathrm{H}_n(X_{k-1})$$

and

$$\mathrm{H}_{n-1}^{\Delta}(X_{k-1}) \longrightarrow \mathrm{H}_{n-1}(X_{k-1})$$

are both isomorphisms. We have now to deal with the two arrows of the form :

$$\mathrm{H}_n^{\Delta}(X_k, X_{k-1}) \longrightarrow \mathrm{H}_n(X_k, X_{k-1}).$$

Let denote by $(\Delta_k^j)_{j \leq j_k}$ the simplices of dimension k of X_k. $\mathrm{H}_n^{\Delta}(X_k, X_{k-1})$ is the group generated by the $[\Delta_k^j)]_j$s for $k = n$ and 0 otherwise. We now compute $\mathrm{H}_n(X_k, X_{k-1})$. Consider the map

$$T : \sqcup_j \Delta_k^k \longrightarrow X_k$$

induced by the individual identifications $\Delta_k^j \to X$ defining the Δ-structure of X_k. T maps $\sqcup_j \partial \Delta_k^k$ onto X_{k-1}, and by definition of the Δ-structure thus defines a homeomorphism

$$T : \sqcup_j \Delta_k^k / \sqcup_j \partial \Delta_k^k \longrightarrow X_k / X_{k-1}.$$

T thus induces a homeomorphism

$$T_* : \mathrm{H}_n(\sqcup_j \Delta_k^k / \sqcup_j \partial \Delta_k^k) \longrightarrow \mathrm{H}_n(X_k / X_{k-1}).$$

By Proposition 50 we have

$$\mathrm{H}_n(\sqcup_j \Delta_k^k / \sqcup_j \partial \Delta_k^k) = \mathrm{H}_n(\sqcup_j \Delta_k^k, \sqcup_j \partial \Delta_k^k) = \oplus_j \mathrm{H}_n(\Delta_k, \partial \Delta_k).$$

By the long exact sequence $\mathrm{H}_n(\Delta_k, \partial \Delta_k)$ is isomorphic to $\mathbb{Z}$ for $k = n$ and 0 otherwise. We can thus conclude that

$$\mathrm{H}_n^{\Delta}(X_k, X_{k-1}) \longrightarrow \mathrm{H}_n(X_k, X_{k-1})$$

is an isomorphism for all $n \geq 0$. Applying the five Lemma we finally obtain that $\mathrm{H}_n^{\Delta}(X_k) \to \mathrm{H}_n(X_k)$ is an isomorphism for all $n \geq 0$. Applying the statement of $k = \dim(X)$ complete the proof of Theorem 52.

2.8.4 Proof of the five Lemma

We first give names to the morphisms in the diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{i_A} & B & \xrightarrow{i_B} & C & \xrightarrow{i_C} & D & \xrightarrow{i_D} & E \\
\downarrow{\scriptstyle f_A} & & \downarrow{\scriptstyle f_B} & & \downarrow{\scriptstyle f_C} & & \downarrow{\scriptstyle f_D} & & \downarrow{\scriptstyle f_E} \\
A' & \xrightarrow{j_A} & B' & \xrightarrow{j_B} & C' & \xrightarrow{j_C} & D' & \xrightarrow{j_D} & E'
\end{array}$$

We want to show that f_C is an isomorphism.

Injectivity Consider $c \in C$ such that $f_C(c) = 0$. Since f_D is an isomorphism, $i_C(c) = 0$. Thus $c \in \mathrm{Im}(i_B)$ and there exists $b \in B$ such that $i_B(b) = c$. Since the diagram commutes and f_B is an isomorphism, $j_B(f_B(b)) = 0$ which implies that there exists $a' \in A'$ such that $j_A(a') = f_B(b)$. Since both f_B and f_A are isomorphisms, $a = f_A^{-1}(a')$ is such that $i_A(a) = b$. Thus b belongs to $\mathrm{Im}(i_A) = \mathrm{Ker}(i_B)$ which implies that $i_B(b) = c = 0$. This proof that f_C is injective.

Surjectivity Consider $c' \in C'$. Let $d' = j_C(c')$ and $d = f_D^{-1}(d')$. Since $d' \in \operatorname{Im}(j_C)$, $j_D(d') = 0$. Since the diagram commutes and f_D and f_E are isomorphisms, $i_D(d) = 0$. This implies that $d \in \operatorname{Im}(i_C)$. Let $c \in C$ such that $i_C(c) = d$. By commutativity of the diagram, both $f_C(c)$ and c' map via j_C onto d'. Thus $c'' = f_C(c)(c')^{-1} \in \operatorname{Ker}(j_C)$. There therefore exists $b' \in B'$ such that $j_B(b') = c''$. Let $b = f_B^{-1}(b')$. By commutativity of the diagram $f_C(i_B(b)) = c''$. This implies that

$$c' = f_C(i_B(b))^{-1} f_C(c) = f_C(i_B(b)^{-1}c) \in \operatorname{Im}(f_C).$$

f_C is thus surjective.

2.9 The Mayer-Vietoris sequence.

Let X be a topological space, and A and B two open subsets of X such that $X = A \cup B$. In this section we show the existence of a long exact sequence

$$\cdots \to \mathrm{H}_n(A \cap B) \to \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \to \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B) \to \cdots$$

The arrow from $\mathrm{H}_n(A \cap B)$ to $\mathrm{H}_n(A) \oplus \mathrm{H}_n(B)$. This arrow is defined by the two inclusions $i^A : A \cap B \longrightarrow A$ and $i^B : A \cap B \longrightarrow B$. Precisely it is the map

$$\begin{array}{ccc} \mathrm{H}_n(A \cap B) & \longrightarrow & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \\ c & \longmapsto & i^A_*(c) \oplus (-i^B_*(c)) \end{array}$$

Note the choice of a minus sign on the second factor that will play a role in proving that the sequence (yet to be defined) is exact.

The arrow from $\mathrm{H}_n(A) \oplus \mathrm{H}_n(B)$ to $\mathrm{H}_n(X)$. Again this one is induced by the inclusions $i_A : A \to X$ and $i_B : B \to X$. Precisely

$$\begin{array}{ccc} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \longrightarrow & \mathrm{H}_n(X) \\ c_A \oplus c_B & \longmapsto & (i_A)_*(c_A) + (i_B)_*(c_B) \end{array}$$

The arrow from $\mathrm{H}_n(X)$ to $\mathrm{H}_{n-1}(A \cup B)$. This one is the least obvious. From Proposition 43 and 44 (applied to the open cover $X = A \cup B$) we get that any chain $c \in \mathcal{C}_n(X)$ such can be written $a + b$ where $a \in \mathcal{C}_n(A)$ and $b \in \mathcal{C}_n(B)$. Assume that $\partial_n(c) = 0$ (so $[c]$ defines a class in homology). Since $\partial_n(a+b) = 0$, the class $d = \partial_n(a) = \partial_n(-b)$ defines a class in $\mathcal{C}_n(A \cap B)$. Since $\partial_{n-1} \circ \partial_n \equiv 0$, we get that $\partial_{n-1}(d) = 0$. We define $\partial([c]) = [\partial_{n-1}(a)] \in \mathrm{H}_{n-1}(A \cap B)$. Last we need to check that $\partial([c])$ is well-defined and does not depend on the choice of a representative of the form $a + b$. Let $c = a' + b'$ be another such representation. $(a' - a) + (b' - b) = 0$ which implies that $(a' - a) \in \mathcal{C}_n(A \cap B)$. Thus $\partial_n(a' - a) = \partial_n(b - b')$ is a coboundary in $\mathcal{C}_{n-1}(A \cap B)$ which implies $[\partial(a - a')]_{\mathrm{H}_{n-1}(A \cap B)} = 0$ and $[\partial_n(a)] = [\partial_n(a')]$. Thus $\partial([c])$ does not depend on the choice of a decomposition $c = a + b$.

Proposition 54. *The sequence*

$$\cdots \to \mathrm{H}_n(A \cap B) \to \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \to \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A \cap B) \to \cdots$$

is a long exact sequence.

Proof

$\mathrm{Im}(i_*^A \oplus -i_*^B) = \mathrm{Ker}((i_A)_* + (i_B)_*)$ A class c in $\mathrm{H}_n(A) \oplus \mathrm{H}_n(B)$ is in the kernel of $(i_A)_* + (i_B)_*)$ if and only if the both its coordinates a and b represent the same class in $\mathrm{H}_n(X)$. A cycle $a \in \mathcal{C}_n(A)$ representing c differs from a cycle $b \in \mathcal{C}_n(B)$ representing c by a boundary d which can be subdivided as to be $d = d_a + d_b$ where d_a is a boundary in $\mathcal{C}_n(A)$ and d_b is a boundary in $\mathcal{C}_n(B)$. Thus, $a - d_a$ and $b + d_b$ represent the same class c in $\mathrm{H}_n(X)$ and thus define a class in $\mathrm{H}_n(A \cap B)$. This is equivalent to the fact that c is in the image of $i_*^A \oplus -i_*^B$.

$\mathrm{Ker}(\partial) = \mathrm{Im}((i_A)_* + (i_B)_*)$ Let c be a class in $\mathrm{H}_n(X)$

2.10 Fundamental group and homology groups of classical spaces

Space	Fundamental group	Homology groups
$\mathbb{R}^n$	$\{1\}$	$\mathrm{H}_n = \{0\}$ for all $n \geq 1$
S^1	$\mathbb{Z}$	$\mathrm{H}_1(S^1) = \mathbb{Z}$
S^n, $n \geq 2$	$\{1\}$	$\mathrm{H}_n(S^n) = \mathbb{Z}$, $\mathrm{H}_k(S^n) = \{0\}$, $1 \leq k < n$
$\mathbb{T}^2 = S^1 \times S^1$	$\mathbb{Z}^2$	$\mathrm{H}_2(\mathbb{T}^2) = \mathbb{Z}$, $\mathrm{H}_1(\mathbb{T}^2) = \mathbb{Z}^2$

2.10.1 Singular homology of the sphere

We use the exact sequence in relative homology together with Proposition 50 to compute the homology groups of the k-sphere. We note the following points:

- an open neighbourhood of $\partial\Delta_k$ in Δ_k deformation retracts onto $\partial\Delta_k$.
- $S^k \simeq \Delta_k/\partial\Delta_k$.

By Proposition 50, we have that for all $n \geq 1$,

$$\mathrm{H}_n(S^k) \simeq \mathrm{H}_n(\Delta_k, \partial\Delta_k).$$

We can thus compute the homology groups of S^k by induction on k. S^0 is a point so we know its homology groups. Assume that we know that $\mathrm{H}_n(S^l) = 0$ for $n \neq l$ and $\mathbb{Z}$ for $n = l$, and that for all $l < k$. Bearing in mind that $\partial\Delta_k \simeq S^{k-1}$ and $\Delta_k/\partial\Delta_k \simeq S^k$, we consider the long exact sequence in homology (for $n \geq 1$)

$$\cdots \to \mathrm{H}_n(\Delta_k) \to \mathrm{H}_n(S^k) \to \mathrm{H}_{n-1}(S^{k-1}) \to \mathrm{H}_{n-1}(\Delta_k) \to \cdots$$

For $n \geq 2$, since $\mathrm{H}_n(\Delta_k) = \mathrm{H}_{n-1}(\Delta_k) = 0$, we obtain an isomorphism

$$\mathrm{H}_n(S^k) \longrightarrow \mathrm{H}_{n-1}(S^{k-1}).$$

For $n = 1$ the map $\mathrm{H}_{n-1}(S^{k-1}) \to \mathrm{H}_{n-1}(\Delta_k)$ is an isomorphism which yields $\mathrm{H}_1(S^k) = 0$. This complete the induction and we obtain

Proposition 55. *For all $n \geq 1$ and $k \geq 1$,*

$$\mathrm{H}_n(S^k) \simeq 0$$

for $n \neq k$ and $\mathrm{H}_n(S^k) \simeq \mathbb{Z}$.

In particular, the map $\mathrm{H}_k^{\Delta}(S^k) \longrightarrow \mathrm{H}_k(S^k)$ is an isomorphism for the Δ-structure on S^k given by the identification $S^k = \partial\Delta_{k+1}$.

Chapter 3

A short introduction to 3-dimensional manifolds

3.1 Orientability

The next four paragraphs cover the material discussed in Lecture 20.

In this paragraph, $n \geq 2$. Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a map that is a homeomorphism onto its image. At any point x, let B_x be a ball around x. We consider the map

$$f_* : \mathrm{H}_n(B_x, B_x \setminus \{x\}) \longrightarrow \mathrm{H}_n(f(B_x), f(B_x) \setminus \{f(x)\}).$$

Proposition 56. $\mathrm{H}_n(B_x, B_x \setminus \{x\})$ *is homeomorphic to* $\mathbb{Z}$.

Proof. This is a consequence of the long exact sequence

$$\mathrm{H}_n(B_x) \longrightarrow \mathrm{H}_n(B_x, B_x \setminus \{x\}) \longrightarrow \mathrm{H}_{n-1}(B_x \setminus \{x\}) \longrightarrow \mathrm{H}_{n-1}(B_x).$$

Since $B_x \setminus \{x\})$ deformation retracts onto S^{n-1}, $\mathrm{H}_{n-1}(B_x \setminus \{x\}) \simeq \mathbb{Z}$. Moreover $\mathrm{H}_n(B_x) = \mathrm{H}_{n-1}(B_x) = 0$ as $B_x \simeq \mathbb{R}^n$ and $n \geq 2$. This implies that the arrow $\mathrm{H}_n(B_x, B_x \setminus \{x\}) \longrightarrow \mathrm{H}_{n-1}(B_x \setminus \{x\})$ is an isomorphism. □

Furthermore, by the inclusion $B_x \subset \mathbb{R}^n$ $f(B_x) \subset \mathbb{R}^n$, we get by applying the excision theorem (to the complement of B_x in $\mathbb{R}^n$) that inclusions induce canonical identifications

$$\mathrm{H}_n(B_x, B_x \setminus \{x\}) \simeq \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

and

$$\mathrm{H}_n(f(B_x), f(B_x) \setminus \{f(x)\}) \simeq \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(x)\}).$$

Finally, using the translation that maps x to $f(x)$ (which preserves the natural orientation of $\mathbb{R}^n$), we get a canonical identification

$$\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(x)\}) \simeq \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

Putting all these identifications together, we obtain that f_* induces a natural map

$$f_* : \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \longrightarrow \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}).$$

Since f is a homeomorphism, it induces a group isomorphism of $\mathbb{Z}$.

Definition 20 (Orientation preserving maps)**.** *We say that f preserves the orientation at x if the map f_* induced on $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ is the identity, and reverses the orientation if it is $-\mathrm{Id}$. We say that f preserves the orientation if it preserves the orientation at every point in U, and reverses the orientation if it reverses the orientation at every point in U.*

Exercise 26. *Show that if f preserves the orientation at one point x and if U is connected, then it preserves the orientation.*

Definition 21 (Orientable manifold)**.** *A topological manifold M is **orientable** if it has an atlas of charts $(\varphi_i, U_i)_{i\in I}$ defining its manifold structure such that all for all i, j such that $U_i \cap U_j \neq \emptyset$,*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \subset \mathbb{R}^n \longrightarrow \varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$$

is orientation-preserving.

Exercise* 27. *Show that a simply-connected manifold is always orientable.*

Exercise* 28. *Show that the Moebius strip, the projective plane $\mathbb{RP}^2$ or the Klein bottle are non-orientable manifolds.*

3.2 2-dimensional manifolds

In this paragraph we briefly review the classification of 2-dimensional manifold (always up to homeomorphism). Recall the definition of the genus g surface Σ_g. We set $\Sigma_0 = S^2$ and for $g \geq 1$, it is obtained from P a $4g$-gon whose sides are labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, \cdots a_g, b_g, a_g^{-1}, b_g^{-1}$ in this order following the negative orientation of the boundary of P, and setting

$$\Sigma_g := P/_\sim$$

where $\sim$ is the equivalence relation consisting in identifying a_i to a_i^{-1} and b_i to b_i^{-1} in such a way that the orientation induced by P on a_i is the opposite as that induced on a_i^{-1}.

Proposition 57. *For all g, Σ_g is a compact, orientable 2-manifold.*

Proof. See Exercise 29

□

The main classification theorem (that we will admit as the proof would take too long) is the following.

Theorem 58 (Classification of compact, oriented surfaces)**.** *A compact, orientable surface is homeomorphic to Σ_g for some non-negative $g \in \mathbb{N}$.*

Exercise 29. *Give a proof of Proposition 57.*

Exercise* 30. *Show that the universal cover of Σ_g is $\mathbb{R}^2$ for $g \geq 1$.*

3.3 3-dimensional manifolds, foreword

Three-dimensional manifolds are some of the most fascinating objects in topology. Their complete classification has been a grail for topologists from the early work of Poincaré in the late 19th century to the visionary programme of geometer Thurston in the early 80s, until this programme was successfully carried out by analysts using PDEs in the early 2000s (earning Perelman the Fields Medal).

Theorem 59 (Poincaré conjecture)**.** *Every compact and simply-connected* 3*-manifold is homeomorphic to* S^3*.*

The goal that we pursue in the next paragraphs is two-fold : we work at establishing a rough classification of 3-manifolds, and use this as a pretext to introduce some methods building on the algebraic topology arsenal that we have built in the previous chapters.

3.4 Some examples

In the case of surfaces, we were able to give a list of 2-manifolds that it is possible to prove to be completely classifying. Although the world of 3-manifolds is far wilder than that of surfaces, we can first review a number of simple constructions (some of which we have already come across).

The sphere S^3. The sphere S^3 is arguably the simplest compact 3-manifold. In particular it is simply-connected (and with the knowledge of the very difficult Poincaré conjecture, it is the only 1-connected three-manifold).

Exercise 31. *Show that S^3 (and S^n for that matter, $n \geq 1$) is orientable.*

The projective space $\mathbb{RP}^3$. The projective space $\mathbb{RP}^3$ is by definition

$$\mathbb{RP}^3 := S^3/_{x\sim -x}.$$

We have seen that $\pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$ and that its fundamental cover is the S^3.

The 3-torus It is by definition

$$\mathbb{T}^3 := S^1 \times S^1 \times S^1.$$

It is our first example of 3-manifold such that

1. with infinite fundamental group ($\mathbb{Z}^3$);
2. covered by $\mathbb{R}^3$.

Products We can consider products of compact 1-dimensional manifolds with 2-dimensional manifolds to get 3-dimensional manifolds. This invites us to consider

$$S^1 \times \Sigma_g$$

for $g \geq 0$. We have

- $\pi_1(S^1 \times \Sigma_g) = \mathbb{Z} \times \pi_1(\Sigma_g)$;
- if $g \geq 1$, the universal cover $\widetilde{S^1 \times \Sigma_g}$ is $\mathbb{R}^3$ (as the universal cover of a product is the product of the universal covers).

Suspensions. We conclude with a slight elaboration on the products that we have just seen. Let $f : \Sigma_g \longrightarrow \Sigma_g$ a homeomorphism. We define the *suspension of* Σ_g by f to be

$$\Sigma_g \ltimes_f S^1 := \Sigma_g \times [0,1]/_{(x,1)\sim(f(x),0)}.$$

Exercise 32. *Show that $\Sigma_g \ltimes_f S^1$ is a manifold.*

Exercise 33. *Show that if f preserves the orientation of Σ_g, $\Sigma_g \ltimes_f S^1$ is orientable.*

Exercise 34. *Show that if $g \geq 1$, the universal cover of $\Sigma_g \ltimes_f S^1$ is $\mathbb{R}^3$.*

Exercise* 35. *Give a presentation of $\pi_1(\Sigma_g \ltimes_f S^1)$.*

3.5 Submanifolds

The next two paragraphs cover the material discussed in Lecture 21.
Let M be a n-dimensional manifold and consider $k \leq n$.

3.5.1 Definition and examples.

Definition 22 (Submanifold)**.** *A subset $N \subset M$ is a submanifold of M if for every $x \in N$ there exists U_x a neighbourhood of x in M and a homeomorphism of $\varphi : U_x \longrightarrow V$ where V is a neighbourhood of 0 in $\mathbb{R}^n$ such that*

- $\varphi(x) = 0$;
- $\varphi(U_x \cap N) = \{0_{n-k}\} \times \mathbb{R}^k$.

In particular, considering the restriction of such maps φ to $U_x \cap N$, N is given the structure of manifold of dimension k. Here are a few examples:

- Any point $\{p\} \subset M$ is a 0-dimensional submanifold.
- Any set of the form $S^1 \times \{p\} \subset \mathbb{T}^2 = S^1 \times S^1$ is a 1-dimensional manifold.
- Consider a suspension $\Sigma_g \ltimes_f S^1$. Any subset of the form $\Sigma_g \times \{t\}$ for $t \in (0,1)$ projects in $\Sigma_g \ltimes_f S^1$ onto a 2-dimensional manifold homeomorphic to Σ_g.

Incompressible surfaces We now turn to the special case of surfaces in 3-manifolds. To some extent, studying surfaces in 3-manifolds can be thought of as another generalisation of the fundamental group : we want to assess the complexity of a 3-manifold by looking at how complicated an embedded surface can look like.

Definition 23 (Incompressible surface)**.** *Let N be a surface in M a 3-manifold. We say that N is **incompressible** if the map induced by the inclusion $i : N \to M$*

$$i_* : \pi_1(N) \longrightarrow \pi_1(M)$$

is injective.

Question 60. *Find a 3-manifold which contains an incompressible Σ_g for infinitely many g.*

Exercise 36. *Show that if $\Sigma_g \subset \mathbb{T}^3$ is an incompressible surface, then $g = 0$ or 1.*

3.6 Connected sums

3.6.1 Definition

We now explain a construction (which works in every dimension $n \geq 2$) to build the "sum" of two manifolds. Let M_1 and M_2 be two n-dimensional manifolds, $n \geq 2$. Let $B_1 \subset M_1$ and $B_2 \subset M_2$ two embedded closed balls in M_1 and M_2 respectively. In particular, the respective boundaries of B_1 and B_2 are both homeomorphic to S^{n-1}.Consider

$$X = (M_1 \setminus \mathring{B}_1) \sqcup (M_2 \setminus \mathring{B}_2).$$

We have inclusions $i_1 : S^{n-1} \longrightarrow \partial B_1$ and $i_2 : S^{n-1} \longrightarrow \partial B_2$ and we define the equivalence relation $\sim$ on X by $x_1 \sim x_2$ if there exists $z \in S^{n-1}$ such that $x_1 = i_1(z)$ and $x_2 = i_2(z)$. Effectively, this equivalence relation identifies points of ∂B_1 and ∂B_2 bijectively, via the natural homeomorphism $i_1^{-1} \circ i_2$. We define

$$M_1 \# M_2 = X/_{\sim}.$$

Definition 24 (Connected sum)**.** *The manifold $M_1 \# M_2$ is called the **connected sum** of M_1 and M_2.*

Exercise* 37. *Show that if both M_1 and M_2 are connected, then $M_1 \# M_2$ does not depend on the choice of balls B_1 and B_2 and is thus well-defined.*

Exercise 38. *Show that for any manifolds M_1 and M_2, $M_1 \# M_2$ is a manifold.*

Proposition 61. *Show that for any manifold M of dimension n, $M \# S^n$ is homeomorphic to M.*

Proof. Left as an exercise.

□

Definition 25 (Prime manifolds)**.** *An n-dimensional manifold M is* ***prime*** *if it is not homeomorphic to a connected sum*

$$M_1 \# M_2$$

where both M_1 and M_2 are not homeomorphic to S^n.

3.6.2 Submanifolds homeomorphic to S^n

We now try to go in the opposite direction, and try to understand when the connected sum operation can be reversed.

In what follows M is an n-dimensional manifold.

- An embedding of S^{n-1} in a manifold M (that is a map $i : S^{n-1} \to M$ that is a homeomorphism onto its image) is said to be *essential* if there is no ball $B_n \subset M$ (that is a subset of M homeomorphic to the ball in $\mathbb{R}^n$) such that

$$\partial B_n = i(S^{n-1}).$$

- An embedding of S^{n-1} is said to be *regular* if it extends to an embedding of $S^{n-1} \times (-1, 1)$.

Definition 26 (Irreducible manifolds)**.** *A manifold M is called irreducible if it has no essential and regular embedding of S^{n-1}.*

3.6.3 Irreducible vs prime manifolds

In this paragraph we reduce our study to the case $n = 3$.

Two manifolds that are prime but not irreducible. Consider the manifold $S^2 \times S^1$. Any subset $S^2 \times \{p\}$ is a regular embedding of S^3 that is essential, S^2 is therefore not irreducible.

Proposition 62. *$S^2 \times S^1$ is prime.*

Proof. Consider any decomposition $S^2 \times S^1 = M_1 \# M_2$. Since $\pi_1(S^2 \times S^1) = \pi_1(M_1 \setminus \mathring{B}_1) * \pi_1(M_2 \setminus \mathring{B}_2) = \mathbb{Z}$. Thus (up to permuting M_1 and M_2), $\pi_1(M_1 \setminus \mathring{B}_1) = 1$ and $\pi_1(M_2 \setminus \mathring{B}_1) = \mathbb{Z}$. Thus $M_1 \setminus \mathring{B}_1$ lifts to the universal cover of $S^2 \times S^1$ which is $S^2 \times \mathbb{R} \simeq \mathbb{R} \setminus \{0\}$. $M_1 \setminus \mathring{B}_1$ is thus realised a compact subset of $\mathbb{R}^3$ whose boundary is an embedded ball. By Alexander's theorem[1], it is an open ball. This implies that $M_1 = S^3$.

□

[1]Alexander's theorem is a standard result about the topology of $\mathbb{R}^3$: it says that any embedded sphere in $\mathbb{R}^3$ bounds a ball.

In analogous way one can consider the manifold defined the following way. Let $f : S^2 \longrightarrow S^2$ induced by the linear map of $\mathbb{R}^3$ defined by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. (This is, up to isotopy, the only orientation-reversing homeomorphism of S^2). Consider the suspension

$$S^2 \ltimes_f S^1.$$

An analogous reasoning to that of the proof of Proposition 62 proves that $S^2 \ltimes_f S^1$ is prime but not irreducible. The following theorem ensures that in dimension 3, $S^2 \times S^1$ and $S^2 \ltimes_f S^1$ are the only two exceptions.

Theorem 63. *Let M a compact 3-manifold. If M is prime then M is irreducible unless it is homeomorphic to $S^1 \times S^2$ or $S^2 \ltimes_f S^1$ where $f : S^2 \longrightarrow S^2$ is the map introduced above.*