Geometry and Groups

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Contents

Warning : these notes are a work in progress and some typos and actual mistakes can be lurking round the corner.

If you notice a typo or a mistake in the notes, please kindly send me an email at s.ghazouani@ucl.ac.uk.

Chapter 1

Euclidean geometry

In this first chapter, we formally construct *Euclidean geometry.* What do we mean by that? We construct a space X which endowed with

1. a way of measuring length between points;
2. a notion of straight line;
3. a way of measuring angles between straight lines;

for which all of Euclid's axioms/postulates/common notions (including the infamous 5^{th} postulate) are satisfied.

1.1 Euclidean space

1.1.1 Basic definitions and notation

Scalar products A *scalar product* on an $\mathbb{R}$-vector space E of dimension d is a bilinear form $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathbb{R}$ which is

- *symmetric* i.e. $\forall \vec{v}, w \in E$, $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$;
- *positive* i.e. $\forall \vec{v} \in E$, $\langle \vec{v}, \vec{v} \rangle \geq 0$;
- *definite* i.e. $\langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = 0$

The **canonical scalar product** on $\mathbb{R}^d$ is by definition

$$\langle \vec{v}, \vec{w} \rangle := \sum_{i=1}^{d} v_i w_i.$$

Definition 1 (Euclidean n-space)**.** *The Euclidean space of dimension $n \in \mathbb{N}$, which we denote by $\mathbb{E}^n$, is the space $\mathbb{R}^n$ together with the canonical scalar product defined above.*

1.1.2 Geodesics, length and distance

Paths A *path* in $\mathbb{E}^n$ is a continuously differentiable map $\gamma : [a,b] \longrightarrow \mathbb{R}^n$, which means that γ is differentiable at all points $t \in [a,b]$, and that the map $t \mapsto \gamma'(t) \in \mathbb{R}^n$ is continuous.

Definition 2 (Length of a path)**.** *The length of a path γ as above is the number*

$$L(\gamma) := \int_a^b ||\gamma'(t)||dt$$

where $||\vec{v}||$ denotes the Euclidean norm of a vector $\vec{v} \in \mathbb{R}^n$.

A path must not be confused with *its image*. There are many different paths which *parametrise* the same curve. Which is why we introduce the following notion.

Let $\gamma : I = [a,b] \longrightarrow \mathbb{R}^d$ be a path, and $\varphi : I \longrightarrow I$ a $\mathcal{C}^1$ map such that $\varphi(a) = a, \varphi(b) = b$ and $\varphi' > 0$. We call a path of the form

$$\gamma \circ \varphi : [a,b) \longrightarrow \mathbb{R}^d$$

a *reparametrisation* of γ. The important fact is that reparametrising a path doesn't change its length.

Proposition 1. *Let γ and φ be as above. Show that*

$$\mathrm{L}(\gamma) = \mathrm{L}(\gamma \circ \varphi).$$

Proof: Exercise. ■

We can now define the distance between two points

Definition 3. *The distance between two points p and $q \in \mathbb{R}^d$ is by definition*

$$d(p,q) := \inf_{\gamma : p \to q} \mathrm{L}(\gamma).$$

We could have directly defined the distance to be

$$d(\mathbf{p},\mathbf{q}) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2} = ||\mathbf{p} - \mathbf{q}||$$

(which we will recover in the next Proposition). The advantage of defining the distance using the infimum over paths is that it *generalises* as soon as one is dealing with a space that has some way of measuring the length of paths. This will be the point of view that we will adopt to define non-Euclidean geometries, and which further down the line is the starting point of Riemannian geometry.

Proposition 2. *The distance d just defined has the following properties: for all $p, q, r \in \mathbb{R}^d$*

1. *(Reflexivity) $d(p,q) = d(q,p)$;*
2. *(Triangle inequality) $d(p,q) \leq d(p,r) + d(r,q)$;*

3. $d(p,q) = 0 \Leftrightarrow p = q$.

Proof: The first two properties are left as an exercise. The third one is a consequence of the next Proposition. ■

Proposition 3. *Let p and q be two points in $\mathbb{R}^n$. We have the following properties*

1.

$$d(p,q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$$

2. If γ is path from p to q, we have $L(\gamma) = d(p,q)$ if and only if γ is a reparametrisation of $t \longmapsto (1-t)p + tq$.

Proof: Consider the path $\gamma_0 : t \longmapsto (1-t)p + tq$. One easily checks that

$$\mathrm{L}(\gamma_0) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$$

as γ_0' is constant equal to $||p - q||$. Therefore

$$d(p,q) \leq \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}.$$

Now consider an arbitrary path $\gamma : [a,b] = I \longrightarrow \mathbb{R}^d$ from p to q. We are going to show that $\mathrm{L}(\gamma) \geq \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}$ (with equality iff γ is a parametrisation of $[p,q]$). We achieve this by decomposing γ into two parts: one that stays on $[p,q]$ and another moving orthogonally to the direction from p to q. Let's do this.

Let $\vec{v_1}$ be a normalised vector in the direction of the line form p to q, $\vec{v_1} := \frac{p-q}{||p-q||}$. We add to $\vec{v_1}$ unit vectors $\vec{v_2}, \cdots, \vec{v_d}$ as to form an orthonormal basis of $\mathbb{R}^d$. Since any vector can be decomposed along a given basis, we write for all t

$$\gamma(t) := p + \alpha_1(t)\vec{v_1} + \alpha_2(t)\vec{v_2} + \cdots + \alpha_d(t)\vec{v_d}.$$

Since γ is continuously differentiable, so are the α_is (this fact is left as an exercise to the reader). The α_is satisfy the following properties

- $\forall i$, $\alpha_i(a) = 0$;
- $\alpha_1(b) = ||p - q||$;
- $\forall i \geq 2$, $\alpha_1(b) = 0$.

Set $\tilde{\gamma} := p + \alpha_1\vec{v_1}$. $\tilde{\gamma}$ is the "part of γ" that stays on $[p,q]$. Since $(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_d})$ is an orthonormal basis, we have

$$\forall t \in [a,b], ||\gamma'(t)|| = \sqrt{\sum_{i=1}^{d} \alpha_i'(t)^2}$$

and

$$||\tilde{\gamma}'(t)|| = \sqrt{\alpha_1'(t)^2} = |\alpha_1'(t)|.$$

We thus have

$$||\gamma'(t)|| \geq ||\tilde{\gamma}'(t)||$$

with equality if and only if $\alpha_2'(t) = \cdots = \alpha_d'(t) = 0$. We therefore have

$$L(\gamma) \geq L(\tilde{\gamma})$$

with equality if and only if $\alpha_2 = \cdots = \alpha_d \equiv 0$. Since $\mathrm{L}(\tilde{\gamma}) = \int_a^b ||\tilde{\gamma}'(t)||dt = \int_a^b |\alpha_1'(t)|dt$, by the triangle inequality we have

$$\mathrm{L}(\tilde{\gamma}) \geq |\int_a^b \alpha_1'(t)dt| = |\alpha(b) - \alpha(a)| = ||p - q||$$

with equality if and only if α_1' does not change sign (and hence is non-negative). We therefore have that $L(\gamma) \geq ||p - q||$ with equality if and only γ is a reparamtrisation of $t \mapsto (1-t)p + tq$ (reparametrisation given by α_1). ■

1.1.3 Angles

To define the notion of *angle*, we assume the knowledge of the functions sine and cosine, as well as their usual properties such as formulae for the sine/cosine of a sum and differentiation rules.

Angle between two vectors

Definition 1 (Angle A). Let $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$. The angle between $\vec{v}$ and $\vec{w}$ is by definition the only $\theta \in [0, \pi]$ such that

$$\frac{\langle \vec{v}, \vec{w} \rangle}{||\vec{v}|| \cdot ||\vec{w}||} = \cos\theta.$$

We give an alternative definition of the angle. If $\vec{v}$ and $\vec{w}$ are two unit vectors, they both belong to the unit circle centred at 0 drawn in the plan that $\vec{v}$ and $\vec{w}$ generate. There are two arcs of this circle joining p the extremity of $\vec{v}$ to q the extremity of $\vec{w}$. Let γ be the shortest of these two arcs, that is the only one that is fully contained in one on the two half-planes cut out by the line through $\vec{v}$.

Definition 2 (Angle B). Let $\vec{v}$, $\vec{w}$ and γ as above. The angle between $\vec{v}$ and $\vec{w}$ is by definition the length of the γ.

The important property is that these two definitions coincide. Writing $\vec{v} = (\cos\theta_v, \sin\theta_v)$ and $\vec{w} = (\cos\theta_w, \sin\theta_w)$ a parametrisation of the arc joining $\vec{v}$ to $\vec{w}$ is $\gamma := \theta \longmapsto (\cos\theta, \sin\theta)$, $\theta_v < \theta < \theta_w$. (up to swapping $\vec{v}$ and $\vec{w}$). According to our second definition of the angle, the angle between $\vec{v}$ and $\vec{w}$ is

$$\int_{\theta_v}^{\theta_w} ||\gamma'(\theta)||d\theta.$$

But $\gamma'(\theta) = (-\sin\theta, \cos\theta)$ therefore $||\gamma'(\theta)|| = 1$, which yields

$$\int_{\theta_v}^{\theta_w} ||\gamma'(\theta)||d\theta = \int_{\theta_v}^{\theta_w} 1 \cdot d\theta = \theta_w - \theta_v.$$

On the other hand, by the first definition of the angle, since $\vec{v}$ and $\vec{w}$ both have norm 1, we have that the angle between $\vec{v}$ and $\vec{w}$ is the only θ such that

$$\cos\theta = \vec{v}\cdot\vec{w}.$$

$\vec{v}\cdot\vec{w} = \cos\theta_v\cos\theta_w + \sin\theta_v\sin\theta_w = \cos(\theta_w - \theta_v)$. We thus have $\theta = \theta_w - \theta_v$; this proves that our two definitions of the angle coincide and can be used interchangeably.

Angle between two lines If two *oriented* lines intersect, the angle between these two lines is the angle between two vectors defining the orientation of these lines.

Angle between two paths crossing If two parametrised paths γ_1 and γ_2 meet at a point p, that is $\exists t_1, t_2$ such that $\gamma_1(t_1) = \gamma_2(t_2) = p$ AND $\gamma_1'(t_1) \neq 0$ and $\gamma_2'(t_2) \neq 0$, then the angle between γ_1 and γ_2 (or their images) is the angle between the two vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$.

1.1.4 Isometries of $\mathbb{E}^d$, first definitions

In this paragraph we study the *group* of isometries of the Euclidean space. In a general context, people call *isometry* a map preserving the geometric structure at hand. The most standard context in which one will encounter isometries is for *metric spaces.*

Definition 4 (Isometries of a metric space)**.** *Let (X, d) be a metric space. A map $T : X \longrightarrow X$ is called an isometry if it is bijective and for any $x, y \in \mathbb{E}^n$, we have $d(T(x), T(y)) = d(x, y)$.*

Specified to the case of $\mathbb{E}^d = (\mathbb{R}^d, d)$, this gives

Definition 5 (Isometries of $\mathbb{E}^d$)**.** *A map $T : \mathbb{E}^d \longrightarrow \mathbb{E}^d$ is called an isometry if it is bijective and for any $x, y \in \mathbb{E}^d$, we have $d(T(x), T(y)) = d(x, y)$.*

Affine isometries. A map $T : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is called *affine* if there exists $A \in \mathrm{M}_d(\mathbb{R})$ and $B \in \mathbb{R}^d$ such that for all $X \in \mathbb{R}^d$, $T(X) = AX + B$.

Definition 6 (Orthogonal group)**.** *The orthogonal group is by definition*

$$\mathrm{O}(d) := \{A \in \mathrm{M}_d(R) \mid {}^tA \cdot A = \mathrm{I}_d\}$$

Matrices in $\mathrm{O}(d)$ are exactly matrices preserving the standard scalar product *i.e.*

$$A \in \mathrm{O}(d) \Leftrightarrow \forall \vec{u}, \vec{v} \in \mathbb{R}^d, \langle A\vec{u}, A\vec{v}\rangle = \langle \vec{u}, \vec{v}\rangle.$$

Proposition 1. An affine map

$$T := X \longmapsto A \cdot X + B$$

defines an isometry $\mathbb{E}^d$ if and only if $A \in \mathrm{O}(d)$.

Technical warning: the proof of this proposition is fairly easy **provided one has a good command of scalar products**, in particular being able to jump from abstract notions such as the norm to the more concrete representation of a scalar product as the product of a vector by the transpose of another one. If it is not the case, you are referred to Appendix A.1. It is recommended that one read its content and **do exercises** to solidify your understanding of these **fundamental** notions.

Proof: We reason by equivalences.

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ d(T(x), T(y)) = d(x, y)$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ d(T(x), T(y))^2 = d(x, y)^2$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ ||T(x) - T(y)||^2 = ||x - y||^2$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ \langle T(x) - T(y), T(x) - T(y)\rangle = \langle x - y, x - y\rangle$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ \langle A(x - y), A(x - y)\rangle = \langle x - y, x - y\rangle$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ \langle Ax, Ax\rangle + \langle Ay, Ay\rangle - 2\langle Ay, Ay\rangle = \langle x, x\rangle + \langle y, y\rangle - 2\langle y, x\rangle$$

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ ||Ax||^2 + ||Ay||^2 - 2\langle Ay, Ax\rangle = ||x||^2 + ||y||^2 - 2\langle y, x\rangle$$

The right-hand side is an infinite list of equalities (one for each pair (x, y)). In particular, when specified to $y = x$, we obtain $||Ax||^2 = ||x||^2||$. From this one easily checks the following equivalence

$$\forall x, y \in \mathbb{E}^d,\ ||T(x)||^2 + ||T(y)||^2 - 2\langle T(y), T(x)\rangle = ||x||^2 + ||y||^2 - 2\langle y, x\rangle \Leftrightarrow \forall x, y \in \mathbb{E}^d,\ \langle Ay, Ay\rangle = \langle y, x\rangle.$$

We thus obtain

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{R}^d,\ \langle Ay, Ax\rangle = \langle y, x\rangle$$

Recall that the scalar product $\langle y, x\rangle$ can be written as the matrix product

$$^t y \cdot x$$

where a vector $x \in \mathbb{R}^d$ is thought of as $(n \times 1)$-matrix. Therefore $\langle Ay, Ax\rangle =^t (Ay) \cdot Ax =^t y \cdot (^t A \cdot A) \cdot x$. We obtain

$$\text{T is an isometry} \Leftrightarrow \forall x, y \in \mathbb{R}^d,\ ^t y \cdot (^t A \cdot A) \cdot x =^t y \cdot x =^t y \cdot \mathrm{I}_d \cdot x$$

where $\mathrm{I}_d \in \mathrm{M}_d(\mathbb{R})$ is the identity matrix. We now use the following result, to be proven below (Lemma 4)

If two $(d \times d)$ matrices M and N are such that $\forall x, y \in \mathbb{R}^d,\ ^t y \cdot M \cdot x =^t y \cdot N \cdot x$ then $M = N$.

Applying this result to $M =^t A \cdot A$ and $N = \mathrm{I}_d$, we obtain

$$\text{T is an isometry} \Leftrightarrow^t A \cdot A = \mathrm{I}_d.$$

■

We conclude with the technical Lemma that we have used in the previous proof.

Lemma 4. *If two $(d \times d)$ matrices M and N are such that $\forall x, y \in \mathbb{R}^d,\ ^t y \cdot M \cdot x =^t y \cdot N \cdot x$ then $M = N$.*

Proof: Let $\vec{e}_i$ be the i-th vector of the canonical basis (that is the vector $(0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 in i-th place). Note that for any matrix M, $^t(\vec{e}_i) \cdot M \cdot \vec{e}_j$ is the coefficient of M is place (i, j). By applying $^t y \cdot M \cdot x =^t y \cdot N \cdot x$ to all pairs $\vec{e}_i, \vec{e}_j$ we obtain that all coefficients of M and N coincide, which proves the lemma. ■

1.1.5 Isometries are affine

In this paragraph we prove the following important theorem, which shows that there are no other examples of isometries that those that we have defined in the previous paragraph.

Theorem 1 (Isometries are affine). Any isometry of $\mathbb{E}^d$ is of the form

$$X \longmapsto A \cdot X + B$$

with $A \in \mathrm{O}(d)$ and $B \in \mathbb{R}^d$.

The proof of this theorem hinges on the following lemma.

Lemma 5 (Characterisation of segments)**.** *A point r belongs the segment $[p, q]$ if and only $d(p, q) = d(p, r) + d(r, q)$.*

Proof: We have $d(p, q)^2 = ||p - q||^2 = \langle p - q, p - q\rangle$. Writing $p - q = (p - r) - (q - r)$ we obtain

$$d(p, q)^2 = \langle (p - r) - (q - r), (p - r) - (q - r)\rangle = 2\langle (p - r), (r - q)\rangle + ||p - r||^2 + ||r - q||^2$$

By the Cauchy-Schwarz inequality,

$$2\langle(p-r),(r-q) \leq 2||p-r|| \cdot ||r-q||$$

with equality if and only if $(p-r)$ are $(r-q)$ positively colinear that is $p-r = \lambda(r-q)$ with $\lambda > 0$. This can be written $r = \frac{\lambda}{1+\lambda}q + \frac{1}{1+\lambda}p$ that is $r \in [p,q]$. Therefore we have

$$d(p,q)^2 = (||p-r||^2 + ||q-r||^2 + 2||p-r|| \cdot ||q-r||) = (||p-r|| + ||q-r||)^2 = (d(p,q) + d(q,r))^2$$

if and only if $r \in [p,q]$. This terminates the proof of our lemma. ■

From this powerful lemma we can extract the following property of isometries.

Proposition 6. *An isometry maps a straight line isometrically onto a straight line.*

Proof: We first start by showing that an isometry maps line segments onto line segments. This follows from the previous lemma: if $r \in [p,q]$, $d(p,q) = d(p,r) + d(r,q)$ which implies $d(T(p),T(q)) = d(T(p),T(r)) + d(T(r),T(q))$ which implies that $T(r)$ is the only point of the segment $[T(p),T(q)]$ that is at distance $d(p,r) = d(T(p),T(r))$ of $T(p)$.

Assume now by contradiction that there there exists a line L such that $T(L)$ is not a line. Then there exists 3 points p,q and r (with $r \in [p,q]$ on L whose images do not lie on a straight line. This contradicts the fact that T maps $[p,q]$ onto $[T(p),T(q)]$. ■

Proof of Theorem 1: We consider T an *arbitrary* isometry (our goal is to eventually prove that T is linear.

We first make the following remark: since the set of maps of the form $X \mapsto AX + B$ with $A \in \mathrm{O}(d)$ (in other words affine isometries) is a group, we have that T is an affine isometry if and only if $U \circ T$ is, for any U an affine isometry. It is thus fine to replace T with any map obtained by composing it with an affine isometry.

Restriction to a map that fixes 0. We can therefore assume that T fixes 0, by composing T with the translation of vector $-T(0)$.

Restriction to a map that fixes $\vec{e_1}, \cdots, \vec{e_d}$. Be careful, in what follow we will be jumping from thinking of elements of $\mathbb{E}^d$ as points to thinking of them as vectors very freely. Recall that $\vec{e_1}, \cdots, \vec{e_d}$ are the vectors of the canonical basis (of which we here think as points in $\mathbb{E}^d$). Since T is an isometry and fixes 0, $||T(\vec{e_i})|| = 1$. By virtue of T being an isometry, the vectors defined by the $T(\vec{e_i})$ for $1 \leq i \leq d$ are pairwise orthogonal. Indeed, since $\vec{e_i}$ and $\vec{e_j}$ are orthogonal for $i \neq j$, we have

$$d(\vec{e_i}, \vec{e_j})^2 = ||\vec{e_i} - \vec{e_j}||^2 = 2$$

therefore

$$d(T(\vec{e_i}, \vec{e_j}))^2 = 2 = ||T(\vec{e_i}) - T(\vec{e_j})||^2 = ||T(\vec{e_i})||^2 + ||T(\vec{e_j})||^2 + 2\langle T(e_i), T(e_j)\rangle.$$

Since we know that $||T(\vec{e_i})|| = 1$, we obtain that for all $i \neq j$, $\langle T(e_i), T(e_j)\rangle = 0$. Thus the set of points $(T(\vec{e_1}), \cdots, T(\vec{e_d})$ in $\mathbb{R}^d$, thought of as a collection of vectors forms an orthonormal basis. We therefore know that there exists a matrix $A \in \mathrm{O}(d)$ such that $A \cdot (T(\vec{e_1}), \cdots, T(\vec{e_d})) = (\vec{e_1}, \cdots, \vec{e_d})$. Therefore, up to composing T by $X \mapsto AX$ we can assume that T fixes the set $\{\vec{0}, \vec{e_1}, \cdots, \vec{e_d}\}$.

An isometry that fixes $\{\vec{0}, \vec{e_1}, \cdots, \vec{e_d}\}$ is the identity. We proof by induction on k that T is the identity on $\mathrm{Span}(\vec{e_1}, \cdots, \vec{e_k})$. The case $k = 1$ is exactly Proposition 6. Assume we know that T is the identity on $\mathrm{Span}(\vec{e_1}, \cdots, \vec{e_k})$. By Proposition 6, T is also the identity on $\mathbb{R} \cdot \vec{e_{k+1}}$. Let x be a point in $\mathrm{Span}(\vec{e_1}, \cdots, \vec{e_{k+1}})$, we can write it as

$$x = x_k + t\vec{e_{k+1}}$$

with $x_k \in \mathrm{Span}(\vec{e_1}, \cdots, \vec{e_k})$ and $t \in \mathbb{R}$. In particular, x in on the line through $2x_k$ and $2t\vec{e_k}$ (as x is middle point between $2x_k$ and $2te_k$). Since $T(2x_k) = 2x_k$ and $T(2t\vec{e_k}) = 2t\vec{e_k}$, T is the identity on the entire line through $2x_k$ and $2te_k$ (again by Proposition 6). In particular $T(x) = x$. This terminates the induction, and the proof that T is the identity map.

Conclusion We have proven that, up to composing T with finitely many affine isometries, it is the identity. Since the set of affine isometries forms a group, T is an affine isometry. This finishes the proof of Theorem 1. ■

1.1.6 Properties of the group of isometries

Definition 3. An isometry of $\mathbb{E}^d$ of the form $X \mapsto A \cdot X + B$ with $A \in \mathrm{O}(d)$ and $B \in \mathbb{R}^d$ is called

- **orientation-preserving** if $\det A = 1$;
- **orientation-reversing** if $\det A = -1$.

Proposition 7. *The maps*

$$\begin{array}{rccc} \mathrm{Lin} & : & \mathrm{Iso}(\mathbb{E}^d) & \longrightarrow & \mathrm{O}(d) \\ & & (X \mapsto AX + B) & \longmapsto & A \end{array}$$

and

$$\begin{array}{rccc} \mathrm{D} & : & \mathrm{Iso}(\mathbb{E}^d) & \longrightarrow & \{1, -1\} \\ & & (X \mapsto AX + B) & \longmapsto & \det A \end{array}$$

are group homomorphisms.

Proof: If $T_1 := X \mapsto A_1X + B_1$ and $T_2 := X \mapsto A_2X + B_2$ we have

$$T_1 \circ T_2(X) = A_1 \cdot A_2X + A_1 \cdot B_2 + B_1.$$

We therefore see that

- $\mathrm{D}(T_1 \circ T_2) = \det(A_1A_2) = \det(A_1)\det(A_1) = \mathrm{D}(T_1)\mathrm{D}(T_2)$;
- $\mathrm{Lin}(T_1 \circ T_2) = A_1A_2 = \mathrm{Lin}(T_1)\mathrm{Lin}(T_2)$.

■

1.2 Classification of isometries of $\mathbb{E}^2$ and $\mathbb{E}^3$

In this paragraph, we give a quantitative description of isometries (sorting them out into different categories: translations, rotations, reflection, ...) and a classification theorem (something saying that the categories that we will have introduced describe all possible isometries).

1.2.1 Different types of isometries of $\mathbb{E}^2$.

Translations We already know translations, they are maps of the form

$$\begin{array}{ccc} \mathbb{E}^d & \longrightarrow & \mathbb{E}^d \\ X & \longmapsto & X + \vec{v} \end{array}.$$

This is by definition the translation of vector $\vec{v}$. Note two important properties of translations: they are

- **orientation-preserving**;
- **without fixed points**.

Rotations A definition we could take for the rotation of angle $\theta \in [0, 2\pi)$ and centre p is the old-fashioned one: take a point q, it lies on the circle centred at p of radius $d(p,q)$, send it to the point r such that the vectors $q - p$ and $r - p$ form an oriented angle θ (this would require that we define the oriented angle, which we haven't done yet, although that wouldn't be too difficult). Another way to go about this is to use algebra and the affine representation of isometries that we have established.

Let R_θ be the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. The map

$$\begin{array}{ccc} \mathbb{E}^2 & \longrightarrow & \mathbb{E}^2 \\ X & \longmapsto & \mathrm{R}_\theta \cdot X + B \end{array}$$

defines the rotation of angle θ and centre $(\mathrm{I}_2 - \mathrm{R}_\theta)^{-1} \cdot B$. In other words, the rotation of angle θ and centre $p \in \mathbb{R}^d$ is given by the formula

$$X \longmapsto \mathrm{R}_\theta \cdot (X - p) + p.$$

We turn it into a formal definition.

Definition 4 (Rotations). A **rotation** of $\mathbb{E}^2$ is by definition any map of the form

$$X \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot X + B$$

with $\theta \in (0, 2\pi)$ and $B \in \mathbb{R}^2$.

- Rotations are **orientation-preserving**;
- Rotations have **a unique fixed point**, their centre.

Reflections We now move to reflections. We are going to define them without giving a forumla first. Let L be a line in $\mathbb{R}^2$. For any point $p \in \mathbb{R}^2$, there is a unique line L' perpendicular to L going through p. Define $T(p)$ to be the unique point on L' such that $[p, T(p)]$ is bisected by L. **We have this way defined a map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, but it is not obvious (yet) that it is an isometry**. This allows us to give the following compact definition of a reflection.

Definition 5 (Reflections). Let L be a straight line in $\mathbb{E}^2$. The **reflection** across L is the unique map $R : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ which maps any point p to the unique point $R(p)$ such that L is the perpendicular bisector of $[p, R(p)]$.

We start by making the simplifying assumption that $0 \in L$. Let $\vec{v}$ be a unit vector spanning L, and $\vec{w}$ a unit vector orthogonal to $\vec{w}$. This way, any point p can be written

$$p = \alpha\vec{v} + \beta\vec{w}.$$

We see this way that the orthogonal projection of p onto L is $\alpha\vec{v}$ and therefore $T(p) = \alpha\vec{v} - \beta\vec{w}$. Noticing that $\beta = \langle p, \vec{w}\rangle$, we obtain the following general formula for the reflection through a line of vector $\vec{v}$:

$$\begin{array}{ccc} \mathbb{E}^2 & \longrightarrow & \mathbb{E}^2 \\ X & \longmapsto & X - 2\langle X, \vec{v}^\perp\rangle\vec{v}^\perp \end{array}$$

where $\vec{v}^\perp$ denotes any (of the two) unit vector perpendicular to $\vec{v}$.

Proposition 8. *Any map of the form $T : X \longmapsto X - 2\langle X, \vec{v}^\perp\rangle\vec{v}^\perp$ is an isometry.*

Proof: It is a straightforward calculation:

$$||T(p) - T(q)|| = ||p - q - 2\langle p - q, \vec{v}^\perp\rangle\vec{v}^\perp||.$$

If $p - q = \alpha\vec{v} + \beta\vec{v}^\perp$, $\beta = \langle p - q, \vec{v}^\perp\rangle$ and therefore $p - q - 2\langle p - q, \vec{v}^\perp\rangle\vec{v}^\perp = \alpha\vec{v} - \beta\vec{v}^\perp$. This way

$$||p - q||^2 = \alpha^2 + \beta^2 = \alpha^2 + (-\beta)^2 = ||p - q - 2\langle p - q, \vec{v}^\perp\rangle\vec{v}^\perp||^2$$

which gives $|T(p) - T(q)|| = ||p - q||$. ■

We now need to verify that reflections across arbitrary lines are also isometries. We use the following key observation

Proposition 9. *If R is a reflection across a line L and T is any isometry of $\mathbb{E}^2$, then $T \circ R \circ T^{-1}$ is the reflection across the line $T(L)$. In particular, the reflection across $T(L)$ is an isometry.*

Proof: A reflection R across a line L is characterised by the following property: for any $p \in \mathbb{E}^2$, L is the perpendicular bisector of $[p, R(p)]$. Since the perpendicular bisector of $[a, b]$ is defined by

$$\{q \mid d(a, q) = d(b, q)\}$$

we have that if T is an isometry, then if L is the perpendicular bisector of $[a, b]$, then $T(L)$ is the perpendicular bisector if $[T(a), T(b)]$.

Apply this fact to $[p, T \circ R \circ T^{-1}(p)]$ and its bisector N. We have that $T^{-1}(N)$ is the bisector of $[T^{-1}(p), R(T^{-1}(p))]$. Since R is the reflection across L, we have that $T^{-1}(N) = L$ which implies that $N = T(L)$. This way, we see that for any p, the perpendicular bisector of $[p, T \circ R \circ T^{-1}(p)]$ is $T(L)$ which proves the fact that $T \circ R \circ T^{-1}$ is the reflection across $T(L)$. ■

- Reflections are **orientation-reversing**;
- the set of fixed points of a reflection is **its reflection line**.

Glides We finish the list of isometries of $\mathbb{R}^2$ with glides. We start with a reflection R across a line L. Let $\vec{v}$ be any vector parallel to L.

Definition 6 (Glides). A glide along L is a map of the form

$$\begin{array}{ccc} \mathbb{E}^2 & \longrightarrow & \mathbb{E}^2 \\ x & \longmapsto & R(x) + \vec{v} \end{array}$$

for any non-zero $\vec{v}$ parallel to L.

- Glides are **orientation-reversing**;
- glides have **no fixed points**.

Here is a little table recapitulating the different properties of the different kind of isometries of $\mathbb{R}^2$ that we have analysed.

Isometry type	**Formula**	**Orientation**	**Fixed point(s)**
Translations	$X \mapsto X + \vec{v}$	Preserving	None
Rotations	$X \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} X + B$	Preserving	A point
Reflections	$X \mapsto \begin{pmatrix} \cos 2\theta & +\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} X + \lambda \cdot (\sin\theta, -\cos\theta)$	Reversing	A line
Glides	$X \mapsto \begin{pmatrix} \cos 2\theta & +\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} X + \vec{v}$	Reversing	None

1.2.2 Classification and the notion of conjugacy

In this paragraph, we state a classification theorem which says in substance that the four types of isometries that we have discussed earlier are the only isometries of $\mathbb{E}^2$ and then discuss the notion of

Theorem 2. Any (non-trivial) isometry of $\mathbb{E}^2$ is either

- a translation;

- a rotation;
- a reflection;
- or a glide.

By *non-trivial* we mean *different from the identity map*. The proof of this theorem is somewhat involved and is the object of the first *non-assessed homework*.

Conjugacy Let $(G, \cdot)$ be a group. We say that two elements g_1 and $g_2 \in G$ are *conjugate* if and only if there exists $h \in G$ such that $g_2 = h \cdot g_1 \cdot h^{-1}$. This is a fairly abstract notion that might not seem very well motivated at this point.

For instance, in the group $\mathrm{GL}(d, \mathbb{R})$ if two matrices A and B are conjugate (*i.e.* $A = PBP^{-1}$) we can say that A and B are the matrices of the *same* linear map $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ but for *two different basis* $\mathcal{B}_1$ and $\mathcal{B}_2$. P is the matrix expressing elements of $\mathcal{B}_1$ as linear combinations of elements of $\mathcal{B}_1$, in other words P is the change-of-basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. The upshot of this discussion is the following.

Two invertible matrices that are conjugate (by another invertible matrix) represent the *same object* (a linear map), they just correspond to different choices of coordinates.

We now push this reasoning to the group $\mathrm{Iso}(\mathbb{E}^2)$. We have already caught a glimpse of what conjugacy does when we looked a reflection, in particular with Proposition 9. We have the following important fact.

Proposition 2. Let T and P be two isometries of $\mathbb{E}^2$. If T is of a certain type (that is either a translation, a rotation, a reflection or a glide) so is $P \circ T \circ P^{-1}$.

This general proposition follows from the following more precise facts, whose proofs are left as an exercise.

1. If T is the translation of vector $\vec{v}$ and $P(X) = AX + B$, $P \circ T \circ P^{-1}$ is the translation of vector $A \cdot \vec{v}$.
2. If T is the rotation of angle θ and centred at p, $P \circ T \circ P^{-1}$ is the rotation of angle θ and centred at $P(p)$.
3. If T is the reflection across the line L, $P \circ T \circ P^{-1}$ is the reflection across the line $P(L)$.
4. If T is the glide of line L and vector $\vec{v}$ (parallel to L); and $P(X) = AX + B$ $P \circ T \circ P^{-1}$ is the glide of line $P(L)$ and vector $A\vec{v}$.

1.2.3 Different types of isometries of $\mathbb{E}^3$

We now investigate isometries of the 3-dimensional space. We proceed as we did for $\mathbb{E}^2$: we identify different types of isometries and then state a classification theorem (ensuring that the types we have identified cover all isometries). In what follows we do not necessarily prove that the maps that we introduce *actually are* isometries, this is left as an exercise to the reader.

Translations Translation (which can be defined in any dimension) are maps of the form

$$X \longmapsto X + B$$

where B is an arbitrary vector of $\mathbb{R}^d$. In other words, translations are isometries whose linear part is the identity matrix.

Rotations

Definition 7. Let L be a line in $\mathbb{E}^3$ and $\theta \in (0, 2\pi)$. The rotation of angle θ with axis L is the map that

- leaves L fixed pointwise;
- acts on any plane P orthogonal to L as the 2-dimensional rotation of angle θ centred at $L \cap P$.

Normal form. Any rotation is conjugate in $\mathrm{Iso}(\mathbb{R}^3)$ to a map of the form

$$X \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot X.$$

Twists

Definition 8. Let L be a line in $\mathbb{E}^3$ and $\theta \in (0, 2\pi)$. A twist of angle θ is a rotation composed with a translation in the direction of its axis.

Normal form. Any twist is conjugate in $\mathrm{Iso}(\mathbb{R}^3)$ to a map of the form

$$X \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot X + \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

for some $t \neq 0$.

Reflections

Definition 9. Let P be a plane. The reflection across P is the map that takes any point $p \notin P$ onto the unique point $R(p)$ such that P is perpendicular bisector if the line segment $[p, R(p)]$ (and which is the identity on P).

Normal form. Any reflection is conjugate in $\mathrm{Iso}(\mathbb{R}^3)$ to a map of the form

$$X \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot X.$$

Rotary-reflections

Definition 10. Let P a plane. A rotatory reflection is the composition between the reflection in P and a rotation of axis L that is perpendicular to P.

Normal form. Any rotatory-reflection is conjugate in $\mathrm{Iso}(\mathbb{R}^3)$ to a map of the form

$$X \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \cdot X$$

for some $t \neq 0$.

Glides

Definition 11. Let P be a plane and $\vec{v}$ a vector parallel to P. A glide is the composition of the reflection in P and the translation of vector $\vec{.}$

Normal form. Any glide is conjugate in $\mathrm{Iso}(\mathbb{R}^3)$ to a map of the form

$$X \longmapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot X + \begin{pmatrix} 0 \\ a \\ b \end{pmatrix}$$

for some $(a, b) \neq (0, 0)$.

1.2.4 Classification of isometries of $\mathbb{E}^3$

We have a classification theorem similar to that for $\mathrm{Iso}(\mathbb{E}^2)$.

Theorem 3. Any (non-trivial) isometry of $\mathbb{E}^3$ is either

- a translation;
- a rotation;
- a twist
- a reflection;
- a rotatory-reflection;
- or a glide.

- **Orientation-preserving** isometries are: translations, rotations and twists.
- **Orientation-reversing** isometries are: reflections, rotatory-reflections and glides.

1.3 Exercises

1.3.1 Length

Exercise 1. *Let $\mathcal{E}$ be the ellipse defined by the equation*

$$ax^2 + by^2 = 1$$

for $a, b > 0$. Show that the length of $\mathcal{E}$ is equal to

$$\frac{4}{\sqrt{a}} \int_0^1 \sqrt{\frac{b^2 + (1-b^2)u^2}{1-u^2}} du.$$

Exercise 2. *Compute the length of the curve in $\mathbb{R}^2$ defined by the equation $y = \frac{1}{2}x^2$ between the point $(0,0)$ and $(1, \frac{1}{2})$.*

Exercise 3. *Let γ be the path defined by*

$$\gamma(t) := (2\cos t - \cos 2t, 2\sin t - \sin 2t)$$

for $t \in [0, 2\pi]$.

1. *Draw the image of γ in $\mathbb{R}^2$.*
2. *Show that the length of γ is equal to 16.*

Exercise 4 (Invariance of the length under reparametrisation)**.** *Let $\varphi : [a,b] \longrightarrow [a,b]$ be a $\mathcal{C}^1$ map such that $\varphi'(t) > 0$ for all t and $\varphi(a) = a$ and $\varphi(b) = b$. Let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a path in $\mathbb{R}^n$. Show that*

$$L(\gamma) = L(\gamma \circ \varphi).$$

Exercise 5. *Recall that a path $\gamma = (\gamma_1, \cdots, \gamma_d) : I \longrightarrow \mathbb{R}^d$ is of class $\mathcal{C}^1$ if any of its components $\gamma_i : I \longrightarrow \mathbb{R}$ is continuously differentiable. Let $(\vec{v_1}, \cdots, \vec{v_d})$ be a basis of $\mathbb{R}^d$ as a vector space.*

1. *Show that there are functions $\alpha_1, \cdots, \alpha_d : I \longrightarrow \mathbb{R}$ such that for all t*

$$\gamma(t) = \sum_{i=1}^{d} \alpha_i(t)\vec{v_i}$$

2. *Show that for all $i \leq d$, the function : $\alpha_i : I \longrightarrow \mathbb{R}$ is continuously differentiable.*

1.3.2 Isometries of $\mathbb{R}^d$

Exercise 6. *By definition,* $\mathrm{GL}(d,\mathbb{R}) := \{M \in \mathrm{M}_d(\mathbb{R}) \mid \det(M) \neq 0\}$*. Show that* $\mathrm{GL}(d,\mathbb{R})$ *is a group.*

Exercise 7. *An affine map of* $\mathbb{R}^d$ *is a map of the form* $\varphi := X \longmapsto AX + B$ *with* A *a* $d \times d$ *matrix and* $B \in \mathbb{R}^d$*. Show that* φ *is a bijection if and only if* $A \in \mathrm{GL}(d,\mathbb{R})$ *(i.e.* $\det A \neq 0$*).*

Exercise 8. *Show that* $\mathrm{O}(d)$ *is a group.*

Exercise 9. *Show that if* $A \in \mathrm{O}(d)$*, then* $|\det A| = 1$*.*

Exercise 10. *Let* $A \in \mathrm{O}(d)$*. Show that if* $\vec{x} \in \mathbb{R}^d \neq 0$ *and* $A \cdot \vec{x} = \lambda\vec{x}$ *then* $\lambda = \pm 1$*.*

Exercise* 11. *Let* $A \in \mathrm{O}(d)$*. Show that if* $\vec{x} \in \mathbb{C}^d \neq 0$ *and* $A \cdot \vec{x} = \lambda\vec{x}$ *then* $|\lambda| = 1$*.*

Exercise 12. *Canonical form of an element of* $\mathrm{O}(d)$

For this exercise, A *is an element of* $\mathrm{O}(d)$*.*

(a) *Show that a polynomial* $P \in \mathbb{R}[X]$ *can be written has a product of polynomials of degree at most* 2*. (You can freely use the fact that a polynomial in* $\mathbb{C}[X]$ *can be written as a product of polynomials of degree* 1*).*

(b) *Show that the linear map* $X \longmapsto A \cdot X$ *either has a (real) eigenvector or stabilises a* 2*-dimensional vector space of* $\mathbb{R}^d$*.*

(c) *Let* E *be a vector subspace of* $\mathbb{R}^d$ *such that* $A \cdot E \subset E$*. Show that* A *preserves the orthogonal of* E*.*

(d) *Show that there is a matrix* $P \in \mathrm{O}(d)$ *such that* $A = P^{-1}DP$ *with* D *of the form*

$$\begin{pmatrix}
1 & 0 & & & \cdots & & & & 0 \\
 & \ddots & & & & & & & \\
\vdots & & 1 & & & & & & \\
 & & & -1 & & & & & \\
0 & & & & \ddots & & & & 0 \\
 & & & & & -1 & & & \\
\vdots & & & & & & (R_{\theta_1}) & & \\
 & & & & & & & \ddots & \\
0 & \cdots & & & 0 & & & & (R_{\theta_k})
\end{pmatrix}$$

where (R_θ) *is a* 2×2 *block of the form* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Exercise 13. *Show that the map from* $\mathrm{Iso}(\mathbb{R}^d)$ *to* $\mathrm{O}(n)$ *which associates to an element in* $\mathrm{Iso}(\mathbb{R}^d)$ *its linear part is a group isomorphism.*

Exercise* 14. *Show that* $\mathrm{Iso}(\mathbb{R}^d)$ *is NOT isomorphic (as a group) to* $\mathrm{O}(d) \times \mathbb{R}^d$*.*

1.3.3 Isometries of $\mathbb{R}^2$

Exercise 15. *Let $T \in \mathrm{Iso}(\mathbb{R}^2)$ be a glide. Show that $T \circ T$ is a translation.*

Exercise* 16. *Show that any translation can be written as the product of two rotations of angle π.*

Exercise 17. *Show that the product of two reflections is either a translation or a rotation depending on whether their reflection axis are parallel or not.*

Exercise 18. *Show that any glide is the product of a reflection and a rotation.*

Exercise 19.** *Show that any isometry of $\mathbb{R}^2$ is a product of at most three reflections.*

Exercise 20. *Let $T_1, T_2 \in \mathrm{SO}(2) \ltimes \mathbb{R}^2 \simeq \mathrm{Iso}^+(\mathbb{R}^2)$ be two orientation-preserving isometries of $\mathbb{R}^2$. Show that*

$$[T_1, T_2] := T_1^{-1} \circ T_2^{-1} \circ T_1 \circ T_2$$

is a translation.

Exercise 21 (Conjugacy)**.** *Give a proof of the following statements.*

1. *If T is the translation of vector $\vec{v}$ and $P(X) = AX + B$, $P \circ T \circ P^{-1}$ is the translation of vector $A \cdot \vec{v}$.*

2. *If T is the rotation of angle θ and centred at p, $P \circ T \circ P^{-1}$ is the rotation of angle θ and centred at $P(p)$.*

3. *If T is the reflection across the line L, $P \circ T \circ P^{-1}$ is the reflection across the line $P(L)$.*

4. *If T is the glide of line L and vector $\vec{v}$ (parallel to L); and $P(X) = AX + B$ $P \circ T \circ P^{-1}$ is the glide of line $P(L)$ and vector $A\vec{v}$*

Exercise 22. *Let $T_1, T_2 \in \mathrm{SO}(2) \ltimes \mathbb{R}^2 \simeq \mathrm{Iso}^+(\mathbb{R}^2)$ be two orientation-preserving isometries of $\mathbb{R}^2$. Assume*

$$[T_1, T_2] := T_1^{-1} \circ T_2^{-1} \circ T_1 \circ T_2$$

is the identity. Show that either

1. *both T_1 and T_2 are translations;*

2. *T_1 and T_2 are rotations with same centre.*

Exercise* 23 (Centraliser subgroup)**.** *Let G be a group and g an element of G. Define*

$$Z(g) := \{h \in G |\ g \cdot h \cdot g^{-1} \cdot h^{-1} = 1\}$$

the commutator of g in G.

1. *Show that $Z(g)$ is a subgroup of G.*

2. *Show that the commutator of a translation in* $\mathrm{Iso}(\mathbb{R}^2)$ *is the subgroup of translations.*

3. *Show that the commutator of a rotation of centre* p *is the set of rotations of centre angle* p*.*

4. *Show that the commutator of a reflection across a line* L *is the union of*

   - *translations in the direction of* L*;*
   - *glides along* L
   - *reflections across any line perpendicular to* L*.*

Exercise 24. *Let* R *be a reflection across a line* L *that forms an angle* θ *with the horizontal line. Show that* R *is of the form*

$$X \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot X + t \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

for some $t \in \mathbb{R}$*.*

Give an equation for L*.*

Exercise* 25. *Let* R *be a glide across a line* L *that forms an angle* θ *with the horizontal line. Show that* R *is of the form*

$$X \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot X + t \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

for some $t \in \mathbb{R}$*.*

Give an equation for L*.*

Exercise* 26. *Let* R *be a glide across a line* L *that forms an angle* θ *with the horizontal line. Show that* R *is of the form*

$$X \longmapsto \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \cdot X + B$$

with $B \in \mathbb{R}^2$*.*

Give an equation for L *and explain how to get the translation factor of* R *as a function of* B *and* θ*.*

1.3.4 Properties of $\mathbb{R}^2$

Exercise 27 (Fifth postulate)**.** *Two straight lines are said to be parallel if they do not intersect. Show that given a line L and a point $p \notin L$, there is a unique line L_p such that*

- *$p \in L_p$;*
- *L_p is parallel to L.*

Exercise* 28 (Uniqueness of triangles)**.** *Let a, b and c be positive real numbers. Show that any two triangles whose sides have respective lengths a, b and c are isometric.*

Exercise* 29.

1. *Show that any regular n-gon can be inscribed in a circle.*
2. *Compute the length of a regular n-gon P_n inscribed in a circle of radius 1.*
3. *Let d_n be the diameter of P_n. Show that (d_n) converges to the length of the circle of radius 1. (That's the difficult part, feel free to move to the next one assuming this result)*
4. *Show that the length of a circle of radius 1 is 2π.*

Exercise 30. *Show that the sum of the internal angles of a triangle is always equal to π.*

Exercise 31 (Intercept Theorem)**.** *Using the definition of the angle between two vectors using the scalar product, give a proof of the Intercept Theorem.*

Chapter 2

Spherical geometry

We introduce in this chapter a new geometry, with its own notion of distance, straight line, angles, etc. which differs from Euclidean geometry. Informally, this geometry is that that is experienced by an observer who lives at the surface of a round sphere in three-dimensional Euclidean space. Effectively, it is more or less the geometry of the Earth. Amongst other things, some of the maths that will be developing in this chapter should answer the following question: why does a plane going from London to San Francisco passes close to the North pole?

2.1 First definitions

The space in which things will take place is the 2-sphere in $\mathbb{R}^3$.

$$\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

2.1.1 Paths and distance

We have already defined paths in $\mathbb{R}^3$, we recall that they are just *continuously differentiable maps* $\gamma : I = [a, b] \longrightarrow \mathbb{R}^3$. We had also defined the length of such a path γ to be

$$L(\gamma) := \int_a^b ||\gamma'(t)||dt.$$

A path on S^2 is now just a path $\gamma : I = [a, b] \longrightarrow \mathbb{R}^3$ such that $\forall t \in I$, $\gamma(t) \in \mathbb{S}^2$. For the rest of this chapter

all considered paths will be paths on $\mathbb{S}^2$.

Definition 12. For any two points $p, q \in \mathbb{S}^2$ we define the **spherical distance** between p and q

$$d_s(p, q) := \inf_{\gamma \text{ path from } p \text{ to } q} L(\gamma)$$

where paths are paths which remain on $\mathbb{S}^2$.

We will get to proving that for arbitrary p and q, $d_s(p,q) = \arccos(p\cdot q)$ where $p\cdot q$ is the scalar product between p and q thought of as vectors in $\mathbb{R}^3$. But we do not know this yet.

2.1.2 Linear isometries

Let A be a matrix in O(3). It is induces a linear map

$$\begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \vec{x} & \longmapsto & A\cdot\vec{x} \end{array}$$

which we know to be an isometry of $\mathbb{E}^3$. In particular it maps points of norm 1 to points of norm 1, in other words

$$A(\mathbb{S}^2) \subset \mathbb{S}^2.$$

Actually, considering the map induced by A^{-1} we see that $A : S^2 \longrightarrow S^2$ is actually a bijection. We prove the following

Proposition 3. Let $A \in \mathrm{O}(3)$. For any $p, q \in \mathbb{S}^2$ we have

$$d_s(Ap, Aq) = d_s(p, q).$$

Proof: See Lectures. ■

This Proposition shows that elements of O(3) are isometries of $\mathbb{S}^2$. In other words, O(3) is a subgroup of $\mathrm{Iso}(\mathbb{S}^2)$.

2.1.3 Distance between two points and great circles

It is now important to answer the two following questions. Given two points p and q;

- what is the shortest way from p to q?
- what is the length of the shortest path(s) from p to q?

Great circles We introduce what we will later show are shortest paths from different points on $\mathbb{S}^2$.

Definition 13. A **great circle** is the intersection between a plane through the origin and $\mathbb{S}^2$.

Note that

1. There is always a great circle through any two given points.

2. Unless $p = -q$, this great circle is unique.

3. If $p = -q$, there is infinitely many such great circles.

You can make an exercise of checking those three points.

Antipodal points If $p = -q$, we say that p and q are *antipodal*.

- p and q are not anitpodal, they cut the unique great circle $\mathcal{C}$ they belong to into two unequal arcs. We define

$$[pq] := \text{the arc of } \mathcal{C} \smallsetminus \{p, q\} \text{ of shortest length.}$$

- If p and q are antipodal, they cut all great circles through p and q in two arcs of equal length (equal to π).

Geodesics A path $\gamma : [a, b] \longrightarrow \mathbb{S}^2$ from $p = \gamma(a)$ to $q = \gamma(b)$ is called a *geodesic* if $\mathrm{L}(\gamma) = d_s(p, q)$. We will often also call the image of γ (which is a subset of $\mathbb{S}^2$) a *geodesic* from p to q.

Theorem 4.
- For any non-antipodal p and $q \in \mathbb{S}^2$, if $\vec{v} \in \mathbb{S}^2$ is a vector orthogonal to p in the plane generated by p and q, then the path

$$t \mapsto \cos t \cdot p + \sin t \vec{v}$$

for $0 \leq t \leq d_s(p, q)$ is a geodesic from p to q. Moreover, this geodesic from p to q (whose image is $[pq]$) is unique.

- If p and $q \in \mathbb{S}^2$ are antipodal, for any $\vec{v} \in \mathbb{S}^2$ orthogonal to p, the path

$$t \mapsto \cos t \cdot p + \sin t \vec{v}$$

for $0 \leq t \leq \pi$ is a geodesic from p to q. There are infinitely many such geodesics.

Proof: To be written. ■

2.1.4 Area

We have so far defined every single object we have been working with (or at the very least hinted at how to make things rigorous if need be). We are going to make a notable exception to this here. The notion of *area* is one that we use frequently in our everyday life, to measure the surface of a flat, of a football pitch or that of a country (note that a country can thought of as lying on the surface of a sphere...). It is however rather difficult to define formally for general shapes. Ok, we are happy to say that the area of a rectangle is the product of the lengths of its sides, but what about the area of the United Kingdom? There is no easy formula for the latter. Unfortunately, there is no easy way around this issue, and defining the notion of area rigorously is the business of *measure theory*, which is a topic that we cannot possibly treat here.

We are going to assume the following thing. To every non-empty open subset U of $\mathbb{S}^2$, there exists a number $A(U) > 0$ which we call the *area of* U and the function

$$A : \{\text{non-empty open subsets of } \mathbb{S}^2\} \longrightarrow \mathbb{R}_{>0}$$

satisfies the following properties

1. (additivity) if $U_1 \cap U_2 = \emptyset$, then $A(U_1 \cup U_2) = A(U_1) + A(U_2)$;
2. (invariance by isometries) for every $T \in \text{Iso}(\mathbb{S}^2)$ and any open set $U \subset \mathbb{S}^2$, $A(T(U)) = A(U)$;
3. (normalisation) $A(\mathbb{S}^2) = 4\pi$.

One of the success of measure theory is to establish the following important structural result.

Theorem 5 (Uniqueness of the area)*.* A function

$$A : \{\text{non-empty open subsets of } \mathbb{S}^2\} \longrightarrow \mathbb{R}_{>0}$$

satisfying the properties above exists and is **unique**.

2.2 Spherical trigonometry

In this paragraph we gather some elements of *spherical* trigonometry. Loosely speaking, trigonometry is the study of triangles and the relationship between their angles, sides and area.

2.2.1 Triangles

We'd better formally define triangles first. We should have a good idea of what they are by now. Consider three points p, q and r that are *non-colinear* (three points are called non-colinear if they do not lie on the same great circle). We make the further assumption that no two of these three points are antipodal. The geodesics $[pq], [qr]$ and $[rp]$ enclose a surface which we call the *triangle* $T = pqr$.

Angle. We must now define what the angle at a vertex of the triangle T is. To do this, we just define the angle between two paths meeting at a point.

Definition 14 (Angle). Let γ_1 and γ_2 two paths meeting at a point p, that is there exists t_1 and t_2 such that $\gamma_1(t_1) = \gamma_2(t_2) = p$. Then the angle between γ_1 and γ_2 at p is the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ in $\mathbb{R}^3$.

This is not fully satisfactory in the sense that if I have a geodesic, I can parametrise it with a path $\gamma(t)$, but $\gamma(-t)$ does the job just as well and it changes the way I compute my angle. But we already knew of this issue, when two curves intersect there are essentially two angles that you can compute (what some might call the *internal* and *external* angle). You just need to make sure you are computing the one you are interested in. **For a triangle, the angle at a vertex p is the angle enclosed by the angular sector that lies *within* the triangle.**

2.2.2 Angle defect formula

We know that in a Euclidean triangle, the sum of the internal angles of a triangle is equal to π. The situation in spherical geometry is a tad more complicated.

Theorem 6 (Angle defect formula). Let T be a spherical triangle with angles α, β and γ. We have

$$\alpha + \beta + \gamma - \pi = \text{Area}(T).$$

Proof: In lectures. ■

2.2.3 The spherical cosine formula

Theorem 7 (Cosine formula). Let T be a spherical triangle with angles α, β and γ and (opposite) sides of respective lengths a, b and c.

2.3 Spherical isometries

We have seen already encountered distance-preserving maps, which we call isometries. We give a formal definition, although this is just a copy-paste of what we've done in the Euclidean case.

Definition 15. The group $\text{Iso}(\mathbb{S}^2)$ is the set of bijections T of $\mathbb{S}^2$ such that for all $p, q \in \mathbb{S}^2$ we have

$$d_s(T(p), T(q)) = d_s(p, q).$$

2.3.1 Isometries are matrices

We have already seen that matrices in O(3) induce an isometry of $\mathbb{S}^2$. In fact, these are the only isometries of $\mathbb{S}^2$.

Theorem 8. Let $T : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be an isometry. There exists $A \in \mathrm{O}(3)$ such that $T(p) = A \cdot p$ for all $p \in \mathbb{S}^2$.

Proof: Consider an arbitrary isometry T of $\mathbb{S}^2$. Extend T to the whole of $\mathbb{R}^3$ using the following formula

$$T(\vec{X}) = ||\vec{X}||T(\frac{\vec{X}}{||\vec{X}||})$$

for $\vec{X} \neq 0 \in \mathbb{R}^3$ and $T(0) = 0$.

An important remark is that since $d_s(p, q) = \arccos(\langle p, q\rangle)$, if T preserves distances on $\mathbb{S}^2$, then for any $p, q \in \mathbb{S}^2$, $\langle Tp, Tq\rangle = \langle p, q\rangle$ (where $\langle \cdot, \cdot\rangle$ is the usual scalar product on $\mathbb{R}^3$). We can now use this property to show that the extension of T to $\mathbb{R}^3$ preserves scalar products. If $\vec{X}$ and $\vec{Y}$ are arbitrary points in $\mathbb{R}^3$

$$\langle T(\vec{X}), T(\vec{Y})\rangle = \langle ||\vec{X}||T(\frac{\vec{X}}{||\vec{X}||}), ||\vec{Y}||T(\frac{\vec{Y}}{||\vec{Y}||})\rangle = ||\vec{X}|| \cdot ||\vec{Y}||\langle T(\frac{\vec{X}}{||\vec{X}||}), T(\frac{\vec{Y}}{||\vec{Y}||})\rangle$$

but by the property above

$$\langle T(\frac{\vec{X}}{||\vec{X}||}), T(\frac{\vec{Y}}{||\vec{Y}||})\rangle = \frac{\langle \vec{X}, \vec{Y}\rangle}{||\vec{X}|| \cdot ||\vec{Y}||}$$

so putting things together we obtain

$$\langle T(\vec{X}), T(\vec{Y})\rangle = \langle \vec{X}, \vec{Y}\rangle.$$

The map T is therefore an isometry of $\mathbb{E}^3$ which fixes 0, it is thus of the form $\vec{X} \longmapsto A \cdot \vec{X}$ with $A \in \mathrm{O}(3)$. This terminates the proof of the theorem. ■

2.3.2 Classification

Like in the case of Euclidean isometry, we must make the distinction between *orientation-preserving* and *orientation-reversing* isometries.

Definition 16. Let $A \in \mathrm{O}(3)$ an isometry of $\mathbb{S}^2$.

- If $\det A = 1$, we say that A is orientation-preserving.
- If $\det A = -1$, we say that A is orientation-reversing.

Orientation-preserving isometries We give a precise description of orientation-preserving isometries. Let P be a plane through 0 and L the line through 0 orthogonal to P.

Fact 1. Given $\theta \in (0, 2\pi)$, there is a unique $A \in \mathrm{SO}(3)$ which

- fixes L pointwise;
- is the rotation of angle θ restricted to P.

We call this isometry A the *rotation of angle θ and of axis L*. If P in an element of $\mathrm{O}(3)$ such that

$$P \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in L$$

then

$$A = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} P^{-1}.$$

The proof of this fact is left as an exercise (it derives from the classification of 3-dimensional Euclidean isometries).

Proposition 4. Let A be a rotation of angle θ.

- A has two antipodal fixed points p and q.
- A stabilises the great circle $\mathcal{C}$ dual to p and q. On this circle, the rotation is just a rotation of angle θ.
- A further stabilises all circle obtained by intersecting a plane parallel to the plane though 0 containing $\mathcal{C}$. On these circles, A is a rotation of angle θ. (Note that these circle are NOT great circles).

Proof: Seen in lectures. ■

Orientation-reversing isometries Let P be a plane through 0 and L the line through 0 orthogonal to P.

Fact 2. Given $\theta \in (0, 2\pi)$, there is a unique $A \in \mathrm{SO}(3)$ which

- stabilises L reversing its orientation;

- is the rotation of angle θ restricted to P.

We call this isometry A the *rotary-reflection of angle θ and of axis L*. If P in an element of $\mathrm{O}(3)$ such that

$$P \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in L$$

then

$$A = P \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} P^{-1}.$$

As for orientation-preserving isometries the proof of this fact is left as an exercise (it derives from the classification of 3-dimensional Euclidean isometries).

Proposition 5. Let A be a rotatory-reflection of angle θ.

- There are two antipodal fixed points p and q such that $A(p) = q$ and $A(q) = p$ ($\{p, q\} = L \cap \mathbb{S}^2$.
- A stabilises the great circle $\mathcal{C}$ dual to p and q. On this circle, the rotation is just a rotation of angle θ.
- A swaps the two hemispheres defined by $\mathcal{C}$.

Proof: Seen in lectures. ■

2.4 Euler formula

We conclude with a short section on the Euler formula.

2.4.1 Triangulations

A *triangulation* of $\mathbb{S}^2$ is the datum of finitely many (filled) triangles $T_1, \cdots, T_m$ such that

1. $\bigcup_{i=1}^m T_i = \mathbb{S}^2$
2. for any $i \neq j$, $T_i \cap T_j$ is a union of whole edges and vertices of both T_i and T_j (in other words if T_i and T_j intersect, it is either along a vertex or *entire* edges of both).

Given a triangulation, we define

- its set of vertices which is the finite subset of points of $\mathbb{S}^2$ which appear as vertex of one of the T_is;
- its set of edges which are the geodesics segments which appear as side of one of the T_is;
- its set of faces which are just the T_is.

2.4.2 The Euler formula

Let $\mathcal{T}$ be a triangulation of $\mathbb{S}^2$. We denote by $e(\mathcal{T})$ the cardinality of its set of edges, $v(\mathcal{T})$ the cardinality of its set of vertices and $f(\mathcal{T})$ its number of faces.

Theorem 9 (Euler formula). Let $\mathcal{T}$ be a triangulation of $\mathbb{S}^2$. Then

$$v(\mathcal{T}) - e(\mathcal{T}) + f(\mathcal{T}) = 2.$$

2.5 Exercises

Exercise 32. *Let x be a point in $\mathbb{S}^2$. Show that there exists a plane $P_x \subset \mathbb{R}^3$ such that for every $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{S}^2$ such that $\gamma(0) = x$, $\gamma'(0) \in P_x$.*

(P_x is called the plane tangent to $\mathbb{S}^2$ at x)

Exercise 33. *For any $0 < \alpha < \pi$ construct a triangle with angles $\alpha, \frac{\pi}{2}$ and $\frac{\pi}{2}$.*

Exercise 34. *Show that any two distinct great circles meet in exactly two antipodal points.*

Exercise 35. *We say that two great circles are orthogonal if they form an angle $\frac{\pi}{2}$ where they meet.*

1. *Show that the above definition makes sense (that is, show that the angle doesn't depend on the choice of one of the two intersection points).*

2. *Construct three different geodesics that are pairwise orthogonal.*

Exercise* 36. *Consider two triples of great circles $(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ and $(\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c)$ like in the previous exercise (any two of the tree elements of each triple intersect orthogonally). Show that there exists T an isometry of $\mathbb{S}^2$ such that $T(\mathcal{C}_1) = \mathcal{C}_a$, $T(\mathcal{C}_2) = \mathcal{C}_b$ and $T(\mathcal{C}_3) = \mathcal{C}_c$.*

Exercise* 37. *Show that in the limit of side lengths all going to 0, the spherical cosine law is equivalent to the Euclidean cosine law.*

Exercise* 38. *Define the circle of radius r about a point $x \in \mathbb{S}^2$ to be the set of points p such that $d(p, x) = r$. Compute the perimeter of any circle of radius r for $r < \pi$.*

Chapter 3

Moebius geometry

In this short chapter, we introduce the Riemann sphere $\hat{\mathbb{C}}$ and the group of admissible transformations defining its geometry.

3.1 Riemann sphere and the stereographic projection

We need to first define the space in which we are going to be working. It is a space that is in bijection with the sphere $\mathbb{S}^2$, but we will want to think of it as the complex plane $\mathbb{C}$ to which we have added a point at infinity.

The stereographic projection Consider the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with its north and south poles $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Consider the plane spanned by the first two coordinates

$$\mathcal{P} := \{(x, y, 0) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}.$$

Note that $\mathcal{P}$ naturally identifies with $\mathbb{C}$

Definition 17 (Stereographic projection(s)). The *(South/North)* stereographic projection is the map which to a point $p \in \mathbb{S}^2 \setminus \{N/S\}$ associates the intersection between the line through p and N/S and $\mathcal{P}$. This defines a map

$$\pi_N : \mathbb{S}^2 \setminus \{N\} \longrightarrow \mathbb{C}$$

(or $\pi_S : \mathbb{S}^2 \setminus \{S\} \longrightarrow \mathbb{C}$).

We now give a formula for the stereographic projection. If p is on $\S^2$, the line through p and N is given by the equation $t(p - N) + N$. In coordinates in is the set of points

$$L_p := \{(tx_p, ty_p, t(z_p - 1) + 1\}.$$

This line intersects $\mathcal{P}$ for t such that $t(z_p - 1) + 1 = 0$ i.e.$t = \frac{1}{1-z_p}$ (note that this formula only make sense if $p \neq N$). We therefore obtain

$$\pi_N(x_p, y_p, z_p) = \frac{1}{1 - z_p}(x_p, y_p).$$

A similar reasoning gives

$$\pi_S(x_p, y_p, z_p) = \frac{1}{1 + z_p}(x_p, y_p).$$

Proposition 6. The stereographic projection π_N (respectively π_S) is a bijection between $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{C}$ (respectively $\mathbb{S}^2 \setminus \{S\}$ and $\mathbb{C}$).

Proof: Exercise. ■

The Riemann sphere Now we see from Proposition 6, the plane $\mathbb{C}$ is in bijection with the sphere minus a point. We therefore define the Riemann sphere as $\mathbb{C}$ to which we have adjoined a point at "infinity", which we denote ∞. Formally, the Riemann sphere $\hat{\mathbb{C}}$ is

$$\hat{\mathbb{C}} = \mathbb{C} \cup \infty.$$

We can extend the stereographic projection π_N to the whole of $\mathbb{S}^2$ by setting $\pi_N(N) = \infty$. This way π_N defines a bijection between $\mathbb{S}^2$ and $\hat{\mathbb{C}}$.

Formally, as *sets*, $\mathbb{S}^2$ and $\hat{\mathbb{C}}$ are the *same* thing (as they are in bijection with each other). The important difference is *how we think about them*, and which *geometry* we associate to each. We have so far defined spherical geometry (which is associated with $\mathbb{S}^2$), we now define *Moebius geometry* which is the geometry associated with $\hat{\mathbb{C}}$.

3.2 Moebius maps, cross-ratio and Moebius circles

3.2.1 Moebius maps

Consider a map of the form

$$z \longmapsto \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$.

- This makes sense if not both c and d are equal to zero.
- If the pair (a, b) is proportional to (c, d), then it is the constant map.
- If $c \neq 0$, the point $-\frac{d}{c}$ is not in the domain of $z \longmapsto \frac{az+b}{cz+d}$, but the limit when $z \to -\frac{d}{c}$ is ∞.
- If $c \neq 0$, the limit when $z \to \infty$ is $\frac{a}{c}$.

If (a,b) is NOT proportional to (c,d), $T : z \longmapsto \frac{az+b}{cz+d}$ naturally extends to $\hat{\mathbb{C}}$ by setting

- $T(-\frac{d}{c}) = \infty$ (with the convention that if $c = 0$ and $d \neq 0$, $\frac{d}{c} = \infty$)
- $T(\infty) = \frac{a}{c}$ (with the convention that if $c = 0$ and $a \neq 0$, $\frac{a}{c} = \infty$).

Note the following important point: (a,b) is not proportional to (c,d) if and only if $ad - bc \neq 0$. We will henceforth use this algebraic characterisation for the well-definedness of T.

Definition 18 (Moebius maps). The map $T : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ thus defined by extending $z \longmapsto \frac{az+b}{cz+d}$ when $ad - bc \neq 0$ is called a Moebius map.

One easily checks that a Moebius map is a bijection of $\hat{\mathbb{C}}$ and that the composition of two Moebius maps is still a Moebius map. Consider a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$, that is $\det A \neq 0$. We associate to it the Moebius map

$$T_A : z \longmapsto \frac{az+b}{cz+d}.$$

The reason we do that is because we have the following Proposition

Proposition 7. Let A and B two matrices in $\mathrm{GL}(2,\mathbb{C})$. We then have

$$T_A \circ T_B = T_{A \cdot B}.$$

Proof: This is straightforward calculation. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$T_A \circ T_B(z) = \frac{a(\frac{\alpha z+\beta}{\gamma z+\delta}) + b}{c(\frac{\alpha z+\beta}{\gamma z+\delta}) + d}$$

$$T_A \circ T_B(z) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}$$

from which we recover $T_A \circ T_B(z) = T_{A \cdot B}(z)$. ■

We now introduce formally the set of Moebius transformations

$$\mathrm{Mob}(\hat{\mathbb{C}}) := \{f : z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{C},\ ad - bc \neq 0\}.$$

This set in endowed with the composition law, and together with this composition we can easily show that we have a group. The only non-trivial thing to show is the existence of an inverse. If $ad - bc \neq 0$, the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, and the inverse of any Moebius map of the form T_A is $T_{A^{-1}}$, by Proposition 7.

Proposition 8 (Matrix representation). The map

$$\begin{array}{ccc} GL(2,\mathbb{C}) & \longrightarrow & Mob(\hat{\mathbb{C}}) \\ A & \longmapsto & T_A \end{array}$$

is a surjective group homomorphism, whose kernel is the subgroup of matrices $\mathbb{C}^* \cdot I_2$.

Proof: Easy consequence of Proposition 7. ■

As an importance consequence, we obtain the isomorphism

$$Mob(\hat{\mathbb{C}}) \simeq GL(2,\mathbb{C})/\mathbb{C}^* = PGL(2,\mathbb{C}).$$

3.2.2 Cross ratio

You might have noted that $Mob(\hat{\mathbb{C}})$ does not preserve any obvious notion of distance. We are going to define the most important quantity in Moebius geometry, the cross-ratio. It is a non-trivial invariant which originates from projective geometry.

Definition 19 (Cross-ratio). The cross-ratio of an *ordered* a 4-tuple of *distinct* points (z_1, z_2, z_3, z_4) in $\hat{\mathbb{C}}$ is the number

$$(z_1 : z_2 : z_3 : z_4) := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \in \hat{\mathbb{C}}$$

For cross-ratios to become an important feature of Moebius geometry, we need to show that there invariant by $Mob(\hat{\mathbb{C}})$.

Proposition 9. For any $A \in Mob(\hat{\mathbb{C}})$ and a 4-tuple of distinct points (z_1, z_2, z_3, z_4) in $\hat{\mathbb{C}}$ we have

$$(Az_1 : Az_2 : Az_3 : Az_4) = (z_1 : z_2 : z_3 : z_4)$$

In order to prove this Proposition we will need the following Lemma.

Lemma 10. *The group* $SL(2,\mathbb{C})$ *is generated by matrices of the form* $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{C}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{C}^*$ *and* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This Lemma is proven in the Appendix.

Proof of Proposition 9 By Lemma 10, it is enough to prove the Proposition for maps represented by matrices of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{C}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{C}^*$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

- Matrices of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ correspond to translations $z \mapsto z + u$, differences $z_i - z_j$ are all individually invariant by translations therefore so is the cross ratio.
- Matrices of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{C}^*$ correspond to maps of the $z \mapsto \lambda^2 z$. Differences $z_i - z_j$ all become $\lambda^2(z_i - z_j)$ under such a map, which again implies that the cross-ratio $(z_1 : z_2 : z_3 : z_4)$ is preserved.
- The matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents the map $T_A = z \mapsto \frac{-1}{z}$. In this case, $z_i - z_j$ becomes $-(z_i^{-1} - z_j^{-1}) = \frac{z_i - z_j}{z_i z_j}$. We get

$$(Az_1 : Az_2 : Az_3 : Az_4) = \frac{(z_3^{-1} - z_1^{-1})(z_4^{-1} - z_2^{-1})}{(z_3^{-1} - z_2^{-1})(z_4^{-1} - z_1^{-1})}$$

which is equal to
$\frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \times \frac{(z_3 z_2)(z_4 z_1)}{(z_3 z_1)(z_4 z_2)} = (z_1 : z_2 : z_3 : z_4)$.

Since every Moebius map corresponds to a matrice in $\mathrm{SL}(2, \mathbb{C})$, this terminates the proof.

3.2.3 Moebius circles

We have defined in the previous paragraph the important preserved quantity of Moebius geometry, the cross-ratio. We now define a class of objects that are somewhat analogous to straight lines in Euclidean and Spherical geometry.

Equation of a circle in the complex plane Let $\mathcal{C}$ be the circle centred at z_0 or radius $r > 0$. A point $z \in \mathbb{C}$ belongs to $\mathcal{C}$ if and only if

$$|z - z_0|^2 = |z|^2 - z_0 \overline{z} - \overline{z_0} z + |z_0|^2 = r.$$

This equation can be rewritten

$$|z|^2 - z_0 \overline{z} - \overline{z_0} z + (|z_0|^2 - r) = 0.$$

We derive from this the following

Proposition 10. Let $u \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. The set of points satisfying the equation

$$|z|^2 + u\overline{z} + \overline{u} z + \alpha = 0 \tag{3.1}$$

is a (possibly empty) circle.

Proof: The set

$$\{|z|^2 + u\overline{z} + \overline{u} z + \alpha = 0\}$$

is the circle of centre $-u$ of radius $\alpha - |u|^2$. ■

Moebius circles and invariance

Definition 20 (Moebius circles). A Moebius circle is either

- a Euclidean circle in $\mathbb{C} \subset \hat{\mathbb{C}}$;
- the union $D \cup \{\infty\}$ where D is a straight line in $\mathbb{C}$.

The important property of Moebius circle is that they are preserved under Moebius maps.

Proposition 11. Let $\mathcal{C}$ be a Moebius circle and let A be a Moebius map. Then $A(\mathbb{C})$ is a Moebius circle.

Proof: In lectures. ∎

3.3 Exercises

Exercise 39. *Show that the stereographic projection π_N is a bijection between $\mathbb{S}^2 \smallsetminus \{N\}$ and $\mathbb{C}$.*

Exercise 40. *What is the range of the map $\pi_S \circ \pi_N^{-1}$? Show that for all z in the its range, $\pi_S \circ \pi_N^{-1}(z) = \frac{1}{\bar{z}}$.*

Exercise* 41. *Show that stereographic projections preserve angles.*

Exercise 42. *Show that for every two pair of distinct points (z_1, z_2) and (w_1, w_2) of points in $\hat{\mathbb{C}}$, there exists a map in $A \in \mathrm{Mob}(\hat{\mathbb{C}})$ such that $(Az_1, Az_2) = (w_1, w_2)$*

Exercise 43. *Show that for every two triples of distinct points (z_1, z_2, z_3) and (w_1, w_2, w_3) of points in $\hat{\mathbb{C}}$, there exists a map in $A \in \mathrm{Mob}(\hat{\mathbb{C}})$ such that $(Az_1, Az_2, Az_3) = (w_1, w_2, w_3)$*

Exercise 44. *A distance on a space X is a function $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ such that*

- $\forall x, y \in X,\ d(x, y) = d(y, x)$;
- *if for some $x, y \in X$, $d(x, y) = 0$, then $x = y$;*
- $\forall x, y, z \in X,\ d(x, y) \leq d(x, z) + d(z, y)$.

Find all the distance on $\hat{\mathbb{C}}$ that are invariant under $\mathrm{Mob}(\hat{\mathbb{C}})$.

Exercise 45. *Show that there is an infinity of Moebius circles through a point $p \in \mathbb{C}$ that are tangent to a given direction through this point.*

Exercise 46. *Show that $\forall z \in \mathbb{C} \smallsetminus \{0, 1\}$*

$$\mathrm{Cr}(z, 1, 0, \infty) = z$$

Exercise 47. *Recall that* $\mathrm{Aff}(\mathbb{C})$ *is the set of maps of the form* $z \mapsto az + b$ *with* $a \in \mathbb{C}^*$ *and* $b \in \mathbb{C}$.

1. *Show that* $\mathrm{Aff}(\mathbb{C})$ *is a subgroup of* $\mathrm{Mob}(\hat{\mathbb{C}})$.
2. *For any* $A \subset \hat{\mathbb{C}}$. *Define* $\mathrm{Fix}(A) := \{f \in \mathrm{Mob}(\hat{\mathbb{C}}),\ \forall a \in A, f(a) = a\}$. *Show that* $\mathrm{Aff}(\mathbb{C}) = \mathrm{Fix}(\{\infty\})$.
3. *What is* $\mathrm{Fix}(\{0, \infty\}$*?*

Exercise 48 (3-transitivity)**.** ***This exercise is the sequel to Exercise 47.*** *Let* z_1, z_2, z_3 *three distinct points of* $\hat{\mathbb{C}}$.

1. *Show that there exists* $f \in \mathrm{Mob}(\hat{\mathbb{C}})$ *mapping* z_1 *to* ∞.
2. *Considering* $\mathrm{Fix}(\{\infty\})$, *show that there exists* $f \in \mathrm{Mob}(\hat{\mathbb{C}})$ *mapping* (z_1, z_2) *to* $(\infty, 0)$.
3. *Show that there exists* $f \in \mathrm{Mob}(\hat{\mathbb{C}})$ *mapping* (z_1, z_2, z_3) *to* $(\infty, 0, 1)$.
4. *Let* w_1, w_2, w_3 *three distinct points in* $\mathbb{C}$. *Show that there exists* $f \in \mathrm{Mob}(\hat{\mathbb{C}})$ *mapping* (z_1, z_2, z_3) *to* (w_1, w_2, w_3).

Exercise 49. *Utilising results from the previous three exercises, show that for any four distinct points in* $\hat{\mathbb{C}}$,

$$\mathrm{Cr}(z_1, z_2, z_3, z_4) \in \mathbb{C} \setminus \{0, 1\}.$$

Exercise 50 (Moebius circles and cross-ratio)**.** *Show that given any three distinct points on* $\hat{\mathbb{C}}$, *there is a unique Moebius circle through these three points.* ***(Hint : first show it for a well-chosen triple of points and then use Exercise 48 to deal with the general case.)***

Exercise* 51 (Moebius circles and cross-ratio)**.** *Show that four points* z_1, z_2, z_3 *and* z_4 *are on the same Moebius circle if and only if* $\mathrm{Cr}(z_1, z_2, z_3, z_4) \in \mathbb{R}$.

Exercise 52. *Let* A *be a subset of* $\hat{\mathbb{C}}$. *Show that if* $|A| \geq 3$, *then* $\mathrm{Fix}(A)$ *is trivial.*

Exercise* 53 (Stabiliser)**.** *Let* A *be a subset of* $\hat{\mathbb{C}}$. *We denote by*

$$\mathrm{Stab}(A) := \{f \in \mathrm{Mob}(\hat{\mathbb{C}}),\ \forall a \in A, f(a), f^{-1}(a) \in A\}.$$

1. *Assume that* A *is finite and* $|A| \geq 3$. *Show that* $\mathrm{Stab}(A)$ *is finite.*
2. *Show that* $\mathrm{Stab}(\mathbb{R} \cup \infty)$ *is* $\mathrm{PSL}(2, \mathbb{R})$, *the image in* $\mathrm{Mob}(\hat{\mathbb{C}})$ *of matrices have real coefficients.*
3. *What is* $\mathrm{Stab}(\mathbb{R})$*?*
4. *Let* $\mathcal{C}$ *be a Moebius circle. Show that* $\mathrm{Stab}(\mathcal{C})$ *is conjugate in* $\mathrm{Mob}(\mathbb{C})$ *to* $\mathrm{PSL}(2, \mathbb{R})$.

Exercise* 54. *Let* $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ *be a map such that for every four distinct points* z_1, z_2, z_3, z_4,

$$\mathrm{Cr}(f(z_1), f(z_2), f(z_3), f(z_4)) = \mathrm{Cr}(z_1, z_2, z_3, z_4).$$

Show that f *is a Moebius map.*

Exercise* 55. *Let* $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ *be a bijection which maps great circles onto great circles. Show that* f *is a Moebius map. Is this still true if* f *is no longer assumed to be a bijection?*

Chapter 4

Hyperbolic geometry

In this chapter we define *hyperbolic geometry*, which is the two dimensional geometry that satisfies all the usual properties of Euclidean geometry *but the fifth postulate*.

4.1 The upper half-plane

We define

$$\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

which we call the *upper half plane*. We will want to think of it as a subset of $\hat{\mathbb{C}}$. Here are some important remarks for what follows.

- $\mathbb{R} \cup \{\infty\}$ is a Moebius circle which cuts $\hat{\mathbb{C}}$ into two halves, one of which being $\mathbb{H}$. It is therefore convenient to think of $\mathbb{R} \cup \{\infty\}$ as the "boundary" circle of $\mathbb{H}$.

4.1.1 Geodesic lines

For $\mathbb{E}$ and $\mathbb{S}^2$ we had first defined how to measure the length of a path, and defined the distance between two points as the infimum length over all paths between these points. A *geodesic line* was then a length minimising path.

For the sake of efficiency, we are going to proceed in a different way here, we first define who the *geodesic lines* are, and then define a way of measuring distances.

Definition 21 (Geodesics). A geodesic (or geodesic line) in $\mathbb{H}$ is either a (Euclidean) half-circle in $\mathbb{H}$ whose centre is in $\mathbb{R}$ or (the intersection with $\mathbb{H}$ of) a vertical line.

You see that geodesic lines are particular cases of Moebius circles. They are exactly those Moebius circles which intersect $\mathbb{R} \cup \{\infty\}$ orthogonally. In particular, to any geodesic one can associate a pair of points on $\mathbb{R} \cup \{\infty\}$:

- for a half-circle, these are the two points at which this half circle intersect $\mathbb{R}$;
- for a vertical line, it is the point at which it intersect $\mathbb{R}$ and ∞.

These points are called *extremities* or *points at infinity* of a geodesic L. We now check one of the first requirement of geodesic lines: that there is one through any two points.

Proposition 12. For any pair of distinct points p and q there is a (unique) geodesic through p and q.

Proof: • If p and q have the same real part, there is a (unique) vertical line through p and q (and furthermore no half-circle in $\mathbb{H}$ centred at a point in $\mathbb{R}$ passes through both p and p).

• If p and q have different real parts, their perpendicular bisector is not horizontal and therefore intersects $\mathbb{R}$ in a unique point r. The half-circle centred at r of radius the *Euclidean* distance between r and p passes through both p and q. The uniqueness comes from the fact that if a half circle passes through p and q, its centred lies on their perpendicular bisector. ■

Finally, we give an explicit equation for geodesics in $\mathbb{H}$.

Proposition 13. A subset of $\mathbb{H}$ is a geodesic if and only if it is equal to

$$\{\alpha|z|^2 + \beta(z+\bar{z}) + \gamma = 0 | z \in \mathbb{H}\}$$

for some $(\alpha, \beta, \gamma) \in \mathbb{R}^3 \smallsetminus \{\vec{0}\}$ such that the set above is non-empty.

Proof: • If $\alpha \neq 0$, $\alpha|z|^2 + \beta(z+\bar{z}) + \gamma = 0$ is the equation of a circle centrer at $\frac{-\beta}{\alpha} \in \mathbb{R}$. Conversely, the equation of a circle of radius r centred at $p \in \mathbb{R}$ is

$$|z-p|^2 = r \Leftrightarrow |z|^2 - p(z+\bar{z}) + |p|^2 - R = 0$$

which is of the above form with $\alpha = 1, \beta = -p$ and $\gamma = |p|^2 - r$.

• If $\alpha = 0$, $\beta(z+\bar{z}) + \gamma = 0$ is the equation of a vertical straight line as it can rewritten

$$\mathrm{Re}(z) = \frac{-\gamma}{\beta}.$$

Furthermore, any vertical line admits an equation of this form. ■

4.1.2 Distance, first definition

We give a first definition of the distance between two points p and q based on the cross-ratio. It might not be the most natural definition (see paragraph 4.2 for equivalent definitions) but it is the most efficient, provided one knows a bit of Moebius geometry.

Definition 22. Let p and q be two points and let a and $b \in \hat{\mathbb{R}}$ be the two extremities of L_{pq} the geodesic through p and q. We define the distance between p and q to be

$$d_{\mathbb{H}}(p,q) := |\log(|\mathrm{Cr}(a,b,p,q)|)|.$$

The advantage with this definition is that it is completely straightforward, you have a ready formula to use. The drawback though, is that it is not clear it satisfies the triangular inequality $d_{\mathbb{H}}(xy) \leq d_{\mathbb{H}}(x,z) + d_{\mathbb{H}}(z,y)$.

Note the following important point: if p tends to either a or b, then $d_h(p,q)$ tends to infinity. In particular, approaching the "boundary" $\hat{\mathbb{R}}$ takes you an arbitrary distance away from any point fixed within $\mathbb{H}$. Finally, it makes the "length" of a geodesic line infinite (we will define this later).

4.1.3 Isometries

We are now going to find explicit examples of isometries of $\mathbb{H}$, and we will again draw inspiration from Moebius geometry. The main theorem of this section is the following theorem.

Theorem 10 (Moebius isometries). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ such that $\det A > 0$. Then the map

$$T_A := z \mapsto \frac{az+b}{cz+d}$$

is an isometry of $\mathbb{H}$. Precisely,

- T_A induces a bijection of $\mathbb{H}$;
- for any $p,q \in \mathbb{H}$, $d_{\mathbb{H}}(T_A(p), T_A(q)) = d_{\mathbb{H}}(p,q)$.

Proposition 14. The Moebius map

$$T : z \longmapsto \frac{az+b}{cz+d}$$

, for a, b, c and $d > 0$ and $ad - bc > 0$ restricted to $\mathbb{H}$ induces a bijection of $\mathbb{H}$.

Proof: If $z \in \mathbb{H}$, $cz + d \neq 0$ so $\frac{az+b}{cz+d} \in \hat{\mathbb{C}}$. To show that $T(z)$ in $\mathbb{H}$, it is enough to show that $\mathrm{Im}(\frac{az+b}{cz+d}) > 0$.

$$\frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}.$$

We therefore have that $\mathrm{Im}(\frac{az+b}{cz+d}) > 0$ if and only if $\mathrm{Im}((az+b)(c\bar{z}+d)) > 0$.

$$(az+b)(c\bar{z}+d) = a|z|^2 + bd + acz + bc\bar{z}.$$

The imaginary part of $(az+b)(c\bar{z}+d)$ is thus $(ad-bc)\mathrm{Im}(z)$ which is positive in our case. Therefore $T(z) \in \mathbb{H}$.

Since T is the restriction of a Moebius map to $\mathbb{H}$ it is injective by virtue of T being injective on the whole of $\hat{\mathbb{C}}$. Since T is a Moebius map, its inverse is given by $z \longmapsto \frac{a'z+b'}{c'z+d'}$

where the matrix $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ is the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\det(A') = \det(A)^{-1}$, $\det(A') > 0$ and a', b', c' and d' are real. Therefore, by the first part of this proof, if $z \in \mathbb{H}$, so is $\frac{a'z+b'}{c'z+d'}$. We can set $y = \frac{a'z+b'}{c'z+d'} \in \mathbb{H}$, and since $\frac{a'z+b'}{c'z+d'}$ is the inverse on $\hat{\mathbb{C}}$ of T, we have $T(y) = z$ which shows that T is surjective. This terminates the proof. ■

Our end goal is to show that the (real) Moebius maps are isometries of $\mathbb{H}$, and when we look at the formula of the distance with seem to be within touching distance of proving it. Indeed

- $d_{\mathbb{H}}(p, q) := |\log(|\mathrm{Cr}(a, b, p, q)|)|$ where a and b are extremities of the geodesic through p and q;
- $d_{\mathbb{H}}(T(p), T(q)) := |\log(|\mathrm{Cr}(u, v, T(p), T(q))|)|$ where u and v are extremities of the geodesic through $T(p)$ and $T(q)$;
- T preserves cross-ratios.

It will therefore be enough to show that $T(a) = u$ and $T(b) = v$ (or maybe $T(a) = v$ and $T(b) = u$).

Proposition 15. Let T be a real Moebius map. Then T maps geodesics onto geodesics.

Proof: We are going to show that maps of the form $z \mapsto z + t$ with $t \in \mathbb{R}$, $z \mapsto \lambda z$ with $\lambda > 0$ or $z \mapsto \frac{-1}{z}$ preserve geodesic lines. Since they generate $\mathrm{PSL}(\mathbb{R})$ as a group, this will show that any map in $\mathrm{PSL}(2, \mathbb{R})$ preserves geodesics.

A map of the form $z \mapsto z + t$ is just a Euclidean translation along the real axis, it therefore maps semi-circles centred at a point on the real axis onto such a semi-circle. Same thing for vertical line.

A map of the form $z \mapsto \lambda z$ maps circle of radius r centred at $p \in \mathbb{R}$ to the circle of radius λr centred at $\lambda p \in \mathbb{R}$, and it maps the vertical line $\{\mathrm{Re}(z) = \alpha\}$ onto the vertical line $\{\mathrm{Re}(z) = \lambda\alpha\}$.

Finally let $\mathcal{G}$ be an arbitrary geodesic defined by

$$\{\alpha|z|^2 + \beta(z + \bar{z}) + \gamma = 0 | z \in \mathbb{H}\}$$

, see Proposition 13 and let $T(z) := \frac{-1}{z}$. Since $z \in \mathbb{H}$, $z \neq 0$, and $z = \frac{-1}{T(z)}$. We therefore get

$$z \in \mathcal{G} \Leftrightarrow \alpha|\frac{1}{T(z)}|^2 - \beta(\frac{1}{T(z)} + \frac{\bar{1}}{T(z)}) + \gamma = 0$$

but since $T(z) \neq 0$ we can multiply the whole equation by $|T(z)|^2 = T(z)T(\bar{z})$ without breaking the equivalence that is

$$z \in \mathcal{G} \Leftrightarrow \alpha - \beta(T(z) + T(\bar{z})) + \gamma|T(z)|^2 = 0.$$

We have therefore obtained that if $z \in \mathcal{G}$, then $T(z)$ is on the geodesic of defined by

$$\{\gamma|z|^2 - \beta(z+\bar{z}) + \alpha = 0 | z \in \mathbb{H}\}.$$

We now conclude that an arbitrary map $T \in \mathrm{PSL}(2,\mathbb{R})$ maps a geodesic $\mathcal{G}$ onto another geodesic. T can be written as a product $T = T_d \circ \cdots \circ T_1$ where the T_is are of the form $z \mapsto z+t$ with $t \in \mathbb{R}$, $z \mapsto \lambda z$ with $\lambda > 0$ or $z \mapsto \frac{-1}{z}$. $T(\mathcal{G}) = T_d \circ \cdots \circ T_1(\mathcal{G})$. But we have proven that $T_1(\mathcal{G}$ is a geodesic, let's call it $\mathcal{G}_1$. Assume we have proven that $\mathcal{G}_k = T_k \circ \cdots \circ T_1(\mathcal{G})$ for $k \leq n$ is a geodesic. Then $T_{k+1} \circ T_k \circ \cdots \circ T_1(\mathcal{G}) = T_{k+1}(\mathcal{G}_k)$ is a geodesic as T_{k+1} is of the form $z \mapsto z+t$, $z \mapsto \lambda z$ or $z \mapsto \frac{-1}{z}$. Therefore $\mathcal{G}_{k+1} = T_{k+1} \circ T_k \circ \cdots \circ T_1(\mathcal{G})$ is a geodesic. By induction, $\mathcal{G}_n = T(\mathcal{G})$ is a geodesic. This completes the proof of this Proposition. ■

We can now go back to the proof of Theorem 10. We were left to prove that if T is a Moebius map with real coefficients and positive determinant, p and q two points lying on a geodesic whose end points are a and b, then the end points u and v of the geodesic through $T(p)$ and $T(q)$ are $T(a)$ and $T(b)$. Thanks to Proposition 15, we know that the image of the geodesic through $T(p)$ and $T(q)$ is $T(\mathcal{G})$ where $\mathcal{G}$ is the geodesic through p and q. But we also know that $\{a,b\} = \mathcal{G} \cap \hat{\mathbb{R}}$, so $\{T(a),T(b)\} = T(\mathcal{G}) \cap T(\hat{\mathbb{R}})$. But since T has real coefficients $T(\hat{\mathbb{R}}) = \hat{\mathbb{R}}$ which means that

$$\{T(a),T(b)\} = T(\mathcal{G}) \cap T(\hat{\mathbb{R}}) = \{u,v\}.$$

So in particular either $T(a) = u$ and $T(b) = v$ or $T(a) = v$ and $T(b) = u$. Since $d_{\mathbb{H}}(T(p),T(q)) := |\log(|\mathrm{Cr}(u,v,T(p),T(q))|)$ we have

$$d_{\mathbb{H}}(T(p),T(q)) = |\log(|\mathrm{Cr}(T(a),T(b),T(p),T(q))|)| \text{ or} |\log(|\mathrm{Cr}(T(b),T(a),T(p),T(q))|)|$$

. Since $\mathrm{Cr}(T(b),T(a),T(p),T(q)) = \frac{1}{\mathrm{Cr}(T(a),T(b),T(p),T(q))}$ we get

$$d_{\mathbb{H}}(T(p),T(q)) = |\log(|\mathrm{Cr}(T(a),T(b),T(p),T(q))|)|$$

which by invariance of cross-ratios under Moebius maps is equal to $|\log(|\mathrm{Cr}(a,b,p,q)|)| = d_{\mathbb{H}}(p,q)$. This completes the proof of Theorem 10.

4.1.4 The algebraic structure of the group of Moebius isometries

We have now seen that a map of the form

$$z \longmapsto \frac{az+b}{cz+d}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$ with $\det A > 0$ induces an isometry of $\mathbb{H}$. Now, is the group of Moebius isometries isomorphic to $\mathrm{GL}(2,\mathbb{R})$? Not quite, as we've seen, if two matrices A and B are colinear (*i.e.* $B = \lambda \cdot A$ with $\lambda \in \mathbb{R}^*$ they induce the same linear map.

Define

$$\mathrm{Mob}(\mathbb{H}) := \{T : z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{R} \text{ and } ad-bc > 0\}.$$

Proposition 11. *The map*

$$\begin{array}{ccc} \mathrm{GL}(2,\mathbb{R}) & \longrightarrow & \mathrm{Mob}(\mathbb{H}) \\ A & \longmapsto & T_A : z \mapsto \frac{az+b}{cz+b} \end{array}$$

is a surjective group homomorphism whose kernel is $\{\lambda \cdot \mathrm{Id} \mid \lambda \in \mathbb{R}^*\} \sim \mathbb{R}^*$*.*

Proof: Surjectivity is by definition. Injectivity is seen the following way: if $z \mapsto \frac{az+b}{cz+b}$ is the identity map then $z = \frac{az+b}{cz+d}$ for all z which implies $cz^2 + (d-a)z + b = 0$ for all z. That implies $c = 0, (d-a) = 0$ and $b = 0$. This proves that $A = a\mathrm{Id}$. ■

In particular, Mob($\mathbb{H}$) is isomorphic to the group

$$\mathrm{GL}(2,\mathbb{R})/\mathbb{R}^* := \mathrm{PGL}(2,\mathbb{R}).$$

4.1.5 Angles

By definition, the hyperbolic angle between two continuously differentiable curves in $\mathbb{H}$ is the same as the Euclidean angle in $\mathbb{R}^2 \sim \mathbb{C}$ within which $\mathbb{H}$ is contained. We have the easy Proposition

Proposition 16. Moebius isometries of $\mathbb{H}$ preserve angles.

Proof: Moebius maps are holomorphic on $\mathbb{H}$ and therefore conformal, they thus preserve angles. ■

4.1.6 Circle at infinity and geodesics

We introduce on last important notion in hyperbolic geometry, which is the *circle at infinity*. Formally the circle at infinity is

$$\partial\mathbb{H} := \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}.$$

It is made of extremities of geodesics, which are points which lie "at infinity" when seen from within $\mathbb{H}$. We have the following important two properties

- for any two points distinct a and $b \in \partial\mathbb{H}$, there exists a *unique* geodesic $\mathcal{G}$ whose extremities are a and b;
- Moebius isometries extend to $\mathbb{H} \cup \partial\mathbb{H}$.

We will sometimes use the notation $\bar{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$, as $\bar{\mathbb{H}}$ is the closure of $\mathbb{H}$ in $\hat{\mathbb{C}}$.

4.2 Distance, alternative definition with length of paths

We give here an alternative route to defining the hyperbolic distance and deriving its basic properties, using **lengths of paths**. For this paragraph, we are just going to forget all the constructions from previous subsections and start from scratch. We just assume the definition of the hyperbolic plane and of real Moebius maps. We follow the approach that we successfully developed to define Euclidean and Spherical geometries.

4.2.1 Length of a path and distance

We start by defining the *hyperbolic* length of an arbitrary path $\gamma : [a, b] \longrightarrow \mathbb{H}$, it is

$$\mathrm{L}_h(\gamma) := \int_a^b \frac{|\gamma'(t)|dt}{\mathrm{Im}(\gamma(t))}.$$

We can thus define the hyperbolic distance between two points p and q as

$$d_h(p, q) := \inf_{\gamma \text{ path from } p \text{ to } q} \mathrm{L}_h(\gamma).$$

That is it, our distance is now defined :) As it is it is pretty useless, as we don't know how to compute it nor do we now which paths (if any!) realise the distance between two given points.

4.2.2 Invariance under $\mathrm{PGL}(2, \mathbb{R})$

A key to understanding the distance d_h is that it is invariant under the action of real Moebius transformations.

Proposition 17. Let γ be a path in $\mathbb{H}$ and $T \in \mathrm{Mob}(\mathbb{H}) \sim \mathrm{PGL}(2, \mathbb{R})$. Then we have

$$\mathrm{L}_h(T \circ \gamma) = \mathrm{L}_h(\gamma).$$

Proof: We compute $\mathrm{L}_h(T \circ \gamma)$. First we compute $(T \circ \gamma)'$. We have

$$(T \circ \gamma)'(t) = \left(\frac{a\gamma(t) + b}{c\gamma(t) + d}\right)' = \frac{a\gamma'(t)(c\gamma(t) + d) - c\gamma'(t)(a\gamma(t) + b)}{(c\gamma(t) + d)^2} = \frac{(ad - bc)\gamma'(t)}{(c\gamma(t) + d)^2}.$$

We now compute $\mathrm{Im}(T \circ \gamma)$. We have

$$T \circ \gamma = \frac{a\gamma + b}{c\gamma + d} = \frac{(a\gamma + b)(c\bar{\gamma} + d)}{|c\gamma + d|^2} = \frac{ac|\gamma|^2 + bd + ad\gamma + bc\bar{\gamma}}{|c\gamma + d|^2}.$$

We derive from this calculation that

$$\mathrm{Im}(T \circ \gamma) = \frac{(ad - bc)\mathrm{Im}(\gamma)}{|c\gamma + d|^2}.$$

This way we get

$$\frac{|(T \circ \gamma)'(t)|}{\mathrm{Im}(T \circ \gamma(t))} = \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))}.$$

This way

$$\mathrm{L}_h(T\circ\gamma)=\int_a^b \frac{|(T\circ\gamma)'(t)|}{\mathrm{Im}(T\circ\gamma(t))}dt=\int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))}dt=\mathrm{L}_h(\gamma).$$ ■

From this Proposition we derive that d_h (defined as $d_h(p,q):=\inf_{\gamma \text{ path from } p \text{ to } q}\mathrm{L}_h(\gamma)$) is invariant under elements of $\mathrm{Mob}(\mathbb{H})$.

Theorem 11. For any $T\in\mathrm{Mob}(\mathbb{H})$, $p,q\in\mathbb{H}$ we have

$$d_h(p,q)=d_h(T(p),T(q)).$$

Proof: Let γ be an arbitrary path from p to q. $T\circ\gamma$ is thus a path form $T(p)$ to $T(q)$. By Proposition 17, we have

$$L_h(\gamma)=\mathrm{L}_h(T\circ\gamma).$$

Since $d_h(p,q)$ is the infimum of the length over all paths from p to q, we immediately get that $d_h(T(p),T(q))\leq d_h(p,q)$ (as for any short path from p to q, we have an equally short path from $T(p)$ to $T(q)$). Applying the same reasoning to $T(p)$, $T(q)$ and T^{-1} we get $d_h(T(p),T(q))\geq d_h(p,q)$. This proves

$$d_h(T(p),T(q))=d_h(p,q).$$ ■

4.3 Classification of isometries of the hyperbolic plane

We have already seen three remarkable classes examples of isometries of $\mathbb{H}$:

1. maps of the form $z\mapsto z+t$ with $t\in\mathbb{R}$;
2. maps of the form $z\mapsto\lambda z$ with $\lambda>0$;
3. maps of the form $z\mapsto\frac{\cos\theta z-\sin\theta}{\sin\theta z+\cos\theta}$ with $\theta\in[0,\pi)$.

Our goal in this paragraph is twofold: to analyse the qualitative behaviour of these and to show that somehow, all other Moebius isometries reduce to these.

4.3.1 Different types of isometries

We start with introducing some terminology.

Definition 23 (Different types of Moebius isometries). A Moebius isometry (different from Id) is called

- *parabolic* if it has no fixed points in $\mathbb{H}$ and a unique one in $\partial\mathbb{H}$;

- *hyperbolic* if it has no fixed points in $\mathbb{H}$ and two in $\partial\mathbb{H}$;
- *elliptic* if it has a (unique) fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$.

We can immediately check which category our examples belong to.

- $z \mapsto z + t$ is parabolic, its unique fixed point is ∞.
- $z \mapsto \lambda z$ is hyperbolic, its two fixed points are 0 and ∞ and consequently it preserves the geodesic through 0 and ∞ (which is just the imaginary axis).
- Finally, $z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$ is elliptic as it fixes i.

Elliptic isometries are direct analogues of Euclidean rotations (as we are going to see this in greater detail in the following paragraphs) whereas hyperbolic and parabolic ones are more analogous to Euclidean translations. But for these two, it is less clear at this point how the analogy should work.

4.3.2 Normalised form of a Moebius isometry and trace

Before we start studying in greater detail our different classes of isometries, we are going to start by clearing the ambiguity on the choice of a matrix representing a Moebius isometry. We have seen that the two matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and λA for $\lambda \neq 0$ represent the same Moebius isometry. A little calculation shows

$$\det(\lambda A) = \lambda^2 \det A$$

and therefore, as $\det A > 0$ there are exactly two choices of $\lambda = \pm(\sqrt{A})^{-1}$ for which λA has determinant 1. Any of these two choice is called a *canonical representative* of the Moebius isometry $T_A = z \mapsto \frac{az+b}{cz+d}$.

Definition 24 (Trace of an isometry). The trace of a Moebius isometry $T = z \mapsto \frac{az+b}{cz+d}$ is

$$\mathrm{Tr}(T) := |\mathrm{Tr}(A)|$$

where A is a canonical representative of T.

Since the two canonical representatives A and B of T are such that $A = -B$, this trace is well-defined. We now explain why the trace has anything to do with the above classification. Let A be a canonical representative of T such that $\mathrm{Tr}(A) \geq 0$ (in practice this representative is unique unless $\mathrm{Tr}(A) = 0$. We try to find the fixed points of T to determine its type. If $z \in \hat{\mathbb{H}}$ is a fixed point of T we have

$$\frac{az+b}{cz+d} = z \Leftrightarrow cz^2 + (d-a)z - b = 0.$$

The discriminant of this degree 2 polynomial equation is

$$\Delta = (d-a)^2 + 4bc = a^2 + d^2 - 2ad + 4bc = (a+d)^2 - 4(ad-bc) = \mathrm{Tr}(A)^2 - 4\det A.$$

Since A is a canonical representative, $\det A = 1$ and we obtain

$$\Delta = |\mathrm{Tr}(A)|^2 - 4.$$

We thus obtain the following.

- If $|\mathrm{Tr}(A)| = 2$, the fixed point equation has a unique real solution and therefore T has a unique fixed point in $\hat{\mathbb{R}} = \partial\mathbb{H}$. In this case, T is **parabolic**.
- If $|\mathrm{Tr}(A)| > 2$, the fixed point equation has a two real solutions and therefore T has a two fixed points in $\hat{\mathbb{R}} = \partial\mathbb{H}$. In this case, T is **hyperbolic**.
- Finally if $|\mathrm{Tr}(A)| < 2$, the fixed point equation has a two complex solutions, one of which with positive imaginary part and the other with negative imaginary part. T has therefore a unique fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$. In this case, T is **elliptic**.

We have therefore proved

Proposition 18. Let T be a Moebius isometry.

- $\mathrm{Tr}(T) = 2 \Leftrightarrow T$ is parabolic.
- $\mathrm{Tr}(T) > 2 \Leftrightarrow T$ is hyperbolic.
- $\mathrm{Tr}(T) < 2 \Leftrightarrow T$ is elliptic.

4.3.3 Parabolic isometries

The prototypical parabolic isometry is the map $z \mapsto z+1$. The following result shows that in some way, it is pretty much the only one.

Proposition 19. Let T be a parabolic isometry. There exists $U \in \mathrm{PGL}(2,\mathbb{R}) = \mathrm{Mob}(\mathbb{H})$ such that

$$U \circ T \circ U^{-1} = (z \mapsto z+1) \text{ or } (z \mapsto z-1).$$

In other words, T is *conjugate* in $\mathrm{PGL}(2,\mathbb{R})$ to either $z \mapsto z+1$ or $z \mapsto z-1$.

Proof: Let $A \in \mathrm{SL}(2,\mathbb{R})$ be a canonical representative of T with positive trace. The characteristic polynomial of A is

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + \det A = X^2 - 2X + 1 = (X-1)^2.$$

We use without proof the following fact form linear algebra: a (2×2)-matrix whose characteristic polynomial is $(X-1)^2$ is conjugate in $\mathrm{SL}(2,\mathbb{R})$ to a matrix $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Now one can check the following easy calculation

$$\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 1 & \alpha h^2 \\ 0 & 1 \end{pmatrix}$$

Since we have $Q \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} Q^{-1} = A$ we obtain $A = R \begin{pmatrix} 1 & \alpha h^2 \\ 0 & 1 \end{pmatrix} R^{-1}$ with $R = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} Q$. h^2 can be chosen as to have $\alpha h^2 = -1, 0$ or 1. $\alpha h^2 = 0$ can be excluded as $A \neq \mathrm{Id}$. We thus obtain, using

$$T_A = T_R \circ P_{\pm} \circ T_R^{-1}$$

with $P_{\pm} = z \mapsto z \pm 1$ that T_A is conjugate to $z \mapsto z \pm 1$. ■

Now that we have this structural result, what can we say about a generic parabolic isometry T, which is conjugate to $z \mapsto z \pm 1$ via another isometry U, $T = U \circ (z \mapsto z \pm 1) \circ U^{-1}$?

- T has unique fixed point, it is $U(\infty)$.
- The horizontal lines $L_a = \{z \in \mathbb{H} \mid \mathrm{Im}(z) = a\}$ are stabilised by $z \mapsto z \pm 1$. Consequently, the curves $\mathcal{C}_a = U(\{z \in \mathbb{H} \mid \mathrm{Im}(z) = a\})$ are stabilised by T.

4.3.4 Hyperbolic isometries

We carry on with our analysis of isometries.

Proposition 20. Let T be a hyperbolic isometry. There exists $U \in \mathrm{PGL}(2,\mathbb{R}) = \mathrm{Mob}(\mathbb{H})$ such that

$$U \circ T \circ U^{-1} = z \mapsto \lambda z$$

In other words, T is *conjugate* in $\mathrm{PGL}(2,\mathbb{R})$ to a map of the form $z \mapsto \lambda z$. Furthermore we have

$$\mathrm{Tr}(T) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}.$$

Proof: As in the proof of Proposition 19, if A is a canonical representative of T with positive trace, we get that the characteristic polynomial of A is

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + 1$$

and since $\mathrm{Tr}(A) > 2$, P_A has two real roots which we call λ and μ. Since $1 = \det A = \alpha\mu$, $\mu = \alpha^{-1}$. Therefore A is diagonalisable, *i.e.* there exists $Q \in \mathrm{SL}(2,\mathbb{R})$ such that

$$QAQ^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

and consequently $T = T_Q \circ (z \mapsto \alpha^2 z) \circ T_Q^{-1}$. Setting $\lambda = \alpha^2$ we get that $\alpha = \sqrt{\lambda}$ (as $\mathrm{Tr}(A) > 0$) and since α is solution of $P_A(\alpha) = 0$ we finally obtain $\sqrt{\lambda}^2 - \mathrm{Tr}(A)\sqrt{\lambda} + 1 = 0$ which transforms into

$$\mathrm{Tr}(T) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}.$$

■

As in the case of parabolic maps, we have a correspondence between fixed points of T and those of $z \mapsto \lambda z$; if $T = U \circ (z \mapsto \lambda z) \circ U^{-1}$, the two fixed points of T are $U(0)$ and $U(\infty)$. Furthermore, T fixed the geodesic whose extremities are $U(0)$ and $U(\infty)$, which is the image under U of the imaginary axis.

Translation length of a hyperbolic isometry We now consider the particular example of $T : z \mapsto \lambda z$, from which we will derive the general case thanks to the above Proposition.

Proposition 12. *The map*

$$\begin{array}{ccc} \mathbb{E}^1 & \longrightarrow & i\mathbb{R} \cap \mathbb{H} \\ t & \longmapsto & ie^t \end{array}$$

is an isometry.

In other word, the geodesic $\mathcal{G} = i\mathbb{R} \cap \mathbb{H}$ with the restriction of the hyperbolic distance is isometric to the real Euclidean line. The proof is straightforward, using the fact that $d_h(ia, ib) = |\log b - \log a|$. Since the isometry $z \mapsto \lambda z$ stabilises $\mathcal{G}$, we look at what it does to it. An easy calculation gives that for any p in $\mathcal{G}$, we have

$$d_h(p, T(p)) = |\log \lambda|.$$

This tells us that restricted to $\mathcal{G}$, $z \mapsto \lambda z$ is nothing but a translation by $|\log \lambda$.

Definition 25 (Translation length). The translation length $l(T)$ of a hyperbolic isometry is $|\log \lambda|$ where λ is such that T is conjugate to $z \mapsto \lambda z$.

This terminology is motivated by the following observation. If T is conjugate to $z \mapsto \lambda z$ by an isometry U (i.e. $T = U \circ (z \mapsto \lambda z) \circ U^{-1}$), we have the following properties:

- the geodesic $U(\mathcal{G})$ is invariant under T;
- on $U(\mathcal{G})$, T is a translation of length $|\log \lambda|$.

Proposition 21. Let T be a hyperbolic Moebius isometry. Then we have

$$l(T) = 2\log(\frac{\mathrm{Tr}(T) + \sqrt{\mathrm{Tr}(T)^2 - 4}}{2}) = 2\cosh^{-1}(\frac{\mathrm{Tr}(T)}{2})$$

Proof: Let A be a canonical representative for T with non-negative trace. The translation length of T is $\log\lambda$, where $\sqrt{\lambda}$ is an eigenvalue of A. Since A has determinant 1, we have

$$\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} = \mathrm{Tr}(A)$$

which implies

$$\sqrt{\lambda}^2 - \mathrm{Tr}(A)\sqrt{\lambda} + 1 = 0.$$

The formula ensues from solving this quadratic equation. ■

We summarise our findings.

- A hyperbolic isometry is characterised by two things: a geodesic $\mathcal{G}$ and a translation length $l > 0$.
- Conversely, given a geodesic $\mathcal{G}$ and a positive number $l > 0$, there exists a unique hyperbolic isometry which fixes $\mathcal{G}$ and translates by l along $\mathcal{G}$.

4.3.5 Elliptic isometries

We conclude with elliptic isometries. They play the role of hyperbolic rotations, and our goal in this section is to explain why. We start with a structural result which will allow us to reduce a lot of question to maps of the form

$$z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}.$$

Proposition 22. Let T be an elliptic isometry. There exists $U \in \mathrm{PGL}(2, \mathbb{R}) = \mathrm{Mob}(\mathbb{H})$ such that

$$U^{-1} \circ T \circ U = z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$$

In other words, T is *conjugate* in $\mathrm{PGL}(2, \mathbb{R})$ to a map of the form $z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$. Furthermore, $\theta \in [0, \pi)$ is determined by

$$\mathrm{Tr}(T) = 2|\cos\theta|.$$

Proof: Again, if A is a canonical representative of T with positive trace, we get that the characteristic polynomial of A is

$$P_A(X) = X^2 - \mathrm{Tr}(A)X + 1.$$

Since $\mathrm{Tr}(A) < 2$ we know that the characteristic polynomial of A factorises as $P_A(X) = (X - e^{i\theta})(X - e^{-i\theta})$. In which case, we know from linear algebra that A can be written as

$$A = P \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} P^{-1}$$

with $P \in \mathrm{SL}(2,\mathbb{R})$ and $\cos\theta > 0$. We get

$$T_A = T_P \circ (z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}) \circ T_P^{-1}.$$

Furthermore, $\mathrm{Tr}(A)$ is the sum of its (complex) eigenvalues which implies $\mathrm{Tr}(A) = e^{i\theta} + e^{i\theta} = 2\cos\theta$. ■

What does the map $z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$ do to $\mathbb{H}$? We know that $T_\theta : z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$ fixes the point $i \in \mathbb{H}$. It therefore maps geodesics through i to geodesics through i. Our ultimate goal is to show that it acts like a rotation about the point i, we therefore try to understand the angle between $\mathcal{G}$ a geodesic through i and its image $T_\theta(\mathcal{G})$.

Consider γ a parametrisation of $\mathcal{G}$ such that $\gamma(0) = i$. $T_\theta \circ \gamma$ is thus a parametrisation of $T_\theta(\mathcal{G})$. The angle between $\mathcal{G}$ and $T_\theta(\mathcal{G})$ is therefore the angle between the *Euclidean* vectors $\gamma'(0)$ and $(T_\theta \circ \gamma)'(0)$. We do the following computation.

$$(T_\theta \circ \gamma)' = (\frac{\cos\theta\gamma - \sin\theta}{\sin\theta\gamma + \cos\theta})' = \frac{\cos\theta\gamma'(\sin\theta\gamma + \cos\theta) - \sin\theta\gamma'(\cos\theta\gamma - \sin\theta)}{(\sin\theta\gamma + \cos\theta)^2}$$

which gives $(T_\theta \circ \gamma)' = \frac{\gamma'}{(\sin\theta\gamma + \cos\theta)^2}$. Now computing at $\gamma(0) = i$ we get $(T_\theta \circ \gamma)'(0) = e^{-2i\theta}\gamma'(0)$. We thus obtain that the (oriented) angle between $\gamma'(0)$ and $(T_\theta \circ \gamma)'(0)$ is -2θ. We reformulate this as the following Proposition.

Proposition 13. *The map $T_\theta : z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}$ rotates any geodesic through i by an angle -2θ. Furthermore, it maps any point p on a half-geodesic from i $\mathcal{H}$ onto the point on $T_\theta(\mathcal{H})$ that is at distance $d_h(i,p)$ from i.*

This Proposition gives a complete description of the map T_θ. From this description, we can describe fully all elliptic maps.

Proposition 14. *The map T be an elliptic Moebius isometry. There exists a point $p_0 \in \mathbb{H}$ and an angle $\alpha \in [0, 2\pi)$ such that T rotates any geodesic through p_0 by an angle α. Furthermore, it maps any point p on $\mathcal{H}$ a half-geodesic from p_0 onto the point on $T(\mathcal{H})$ that is at distance $d_h(p_0, p)$ from p_0.*

This motivates the following alternative terminology for elliptic Moebius isometries.

> *Definition* 26. Rotation The rotation of centre $p_0 \in \mathbb{H}$ of angle α is the elliptic Moebius isometry which fixes p_0 and which rotates any geodesic through p_0 by an angle α.

4.3.6 Isometries are (almost) Moebius

Hyperbolic reflections One can check that the map $R := z \mapsto -\bar{z}$ is the reflection in the geodesic $\mathcal{G} := i\mathbb{R} \cap \mathbb{H}$:

- it is an isometry;
- it fixes $\mathcal{G}$ pointwise;
- maps a point $p \notin \mathcal{G}$ onto a point $T(p)$ such that the geodesic $(p, R(p))$ intersects $\mathcal{G}$ orthogonally.

Now let A be a Moebius transformation. By definition,

$$U := A \circ R \circ A^{-1}$$

is the reflection in $A(\mathcal{G})$. One easily checks that

- it is an isometry;
- it fixes $A(\mathcal{G})$ pointwise;
- maps a point $p \notin A(\mathcal{G})$ onto a point $U(p)$ such that the geodesic $(p, U(p))$ intersects $\mathcal{G}$ orthogonally.

Orientation-reversing isometries We have the following Proposition.

> *Proposition* 23. Any map of the form
>
> $$z \longmapsto \frac{a\bar{z}+b}{c\bar{z}+d}$$
>
> with a, b, c and d real and $ad - bc < 0$ induces an isometry of $\mathbb{H}$.

Proof: Exercise. ■

Maps of the above form are called *Moebius orientation-reversing* isometries of $\mathbb{H}$.

The whole isometry group of $\mathbb{H}$.

> *Theorem* 12. Moebius isometries and the reflection $z \mapsto -\bar{z}$ generate the whole isometry group of $\mathbb{H}$.

In particular, an isometry is either a Moebius isometry or a Moebius orientation-reversing isometry. We leave the proof of this Theorem to the motivated reader. It follows the same lines as the proof of the equivalent theorem in Euclidean geometry.

4.4 Hyperbolic trigonometry

4.4.1 Area and triangles

Definition 27 (Hyperbolic triangles). Let p, q, r be points in $\hat{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$. The triangle $T(p, q, r) = [p, q] \cup [q, r] \cup [r, p]$ where $[a, b]$ is the line segment between a and b (if $a \in \partial\mathbb{H}$, $[a, b]$ is the half geodesic from b to a and if both $a, b \in \partial\mathbb{H}$, it is the entire geodesic).

The angle at a vertex p of a triangle is just the angle between the two line segments $[p, q]$ and $[p, r]$ at p. We use the convention that if $p \in \partial\mathbb{H}$, the angle at p is 0.

Definition 28 (Area). Let U be an open subset of $\mathbb{H}$ (which we call a *domain*). The *hyperbolic area* of U is by definition

$$\mathrm{A}_h(U) = \int_U \frac{dxdy}{y^2}.$$

An important property of the area is that it is invariant by Moebius isometries. Precisely, if $T \in \mathrm{Mob}(\mathbb{H})$ and $U \subset \mathbb{H}$ is a domain we have

$$\mathrm{A}_h(T(U)) = \mathrm{A}_h(U).$$

The proof of this fact is a straightforward application of the change of variable formula involving the Jacobian of T. It is left as an exercise.

4.4.2 Gauss-Bonnet formula (angle defect)

Theorem 13 (Gauss-Bonnet). Let T be a hyperbolic triangle with vertices $p, q, r \in \hat{\mathbb{H}}$ and angles α, β and γ. Then we have

$$\mathrm{A}_h(T) = \pi - (\alpha + \beta + \gamma).$$

Proof: We give a proof of the Gauss-Bonnet formula.

Step 1, $r = \infty$ and p, q on a certain geodeisc We first start with a triangle $T = T(p, q, r)$ where $p, q \in \hat{\mathbb{H}}$ and p and q are arbitrary points on the geodesic from -1 to 1, and $r = \infty$.

We denote by α the angle at p and β the angle at q. The equation for the triangle T is

$$\{z = x + iy \in \mathbb{H} \mid x \in [\cos(\pi - \alpha), \cos\beta], y \geq \sqrt{1 - x^2}\}.$$

This can be seen making the following observations

- if θ is the angle between the Euclidean segment $[-1, 0]$ and $[0, x + iy]$, $x + iy$ is on the geodesic segment (which is the arc of the Euclidean circle of radius 1 centred at 0) between p and q if and only is $(x, y) = (cos\theta, \sin\theta)$ with $\theta \in [\beta, \pi - \alpha]$.

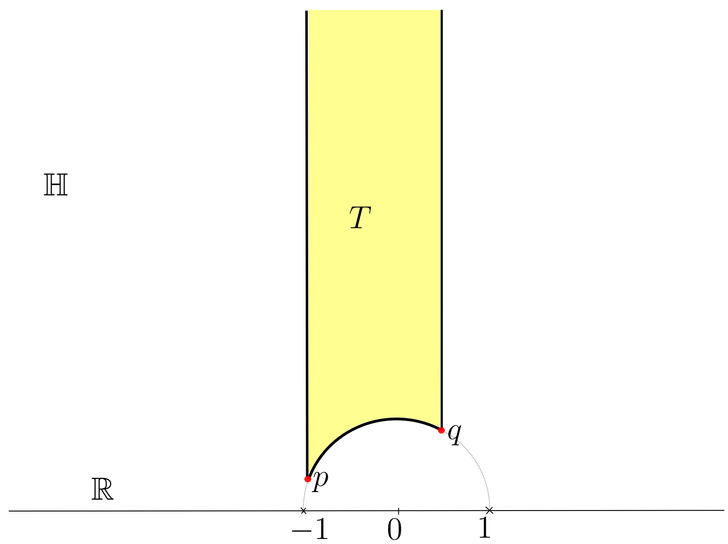

Figure 4.1: A triangle with one vertex at ∞

- (x, y) is in T if the vertical line through $(x,)$ intersects the geodesic segment between p and q, which is equivalent to $y \geq \sin\theta$ and $x = \cos\theta$ with $\theta \in [\beta, \pi - \alpha]$.

From this equation for T we can compute $\mathrm{A}_h(T)$.

$$\mathrm{A}_h(T) = \int_T \frac{dxdy}{y^2} = \int_{\cos(\pi-\alpha)}^{\cos\beta} (\int_{y=\sqrt{1-x^2}}^{+\infty} \frac{dy}{y^2})dx$$

but $\int_{y=\sqrt{1-x^2}}^{+\infty} \frac{dy}{y^2}) = [-\frac{1}{y}]_{\sqrt{1-x^2}}^{\infty} = \frac{1}{\sqrt{1-x^2}}$ from which we obtain

$$\mathrm{A}_h(T) = \int_{\cos(\pi-\alpha)}^{\cos\beta} \frac{dx}{\sqrt{1-x^2}}.$$

Doing the change of variable $x = \cos\theta$ we obtain $dx = -\sin\theta d\theta$ and $\sqrt{1-x^2} = \sin\theta$ and sin is positive on the interval on which we are integrating. Thus we get

$$\mathrm{A}_h(T) = -\int_{\beta}^{\pi-\alpha} \frac{-\sin\theta}{\sin\theta} d\theta = -(\beta - (\pi - \alpha)) = \pi - (\alpha + \beta).$$

Step 2, $r = \infty$ We just observe that using an isometry of the form $z \mapsto \lambda z + t$, we can map any triangle with $r = \infty$ to a triangle like in Step 1. Since $z \mapsto \lambda z + t$ preserves both angles and area, we have $\mathrm{A}_h(T) = \pi - (\alpha + \beta)$ as well.

Step 3, $r \neq \infty$ If none of the vertices are ∞, consider the three triangles $T_1 = (\infty, p, r)$, $T_2 = (\infty, q, r)$ and $T_3 = (\infty, p, q)$. One of the three triangles (say T_3) is the union of the two others and T, i.e.

$$T_3 = T_1 \cup T_2 \cup T.$$

We denote by

- α, β and γ the angles of T at p, q and r respectively;
- α' and γ_1 the angles of T_1 at p and r respectively;
- β' and γ_2 the angles of T_2 at q and r respectively;

See Figure 4.2 for more detail about the configuration.

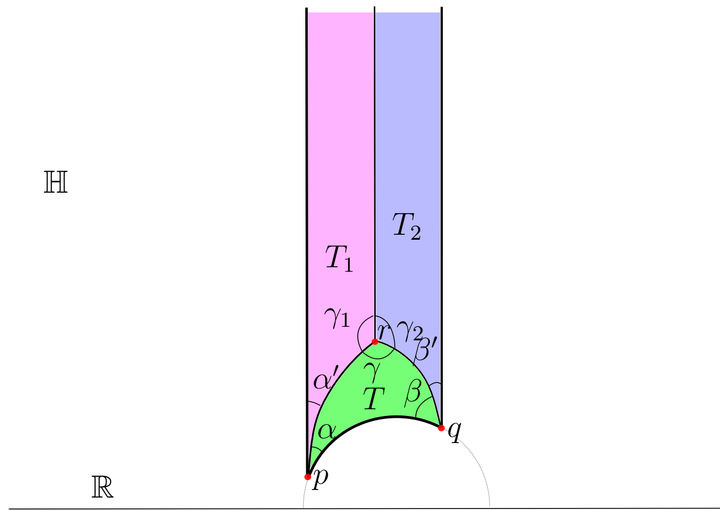

Figure 4.2: T, T_1 and T_2.

Now we have

$$\mathrm{A}_h(T_3) = \mathrm{A}_h(T) + \mathrm{A}_h(T_1) + \mathrm{A}_h(T_2).$$

By Step 2, we have

- $\mathrm{A}_h(T_3) = \pi - (\alpha + \alpha' + \beta + \beta')$;
- $\mathrm{A}_h(T_1) = \pi - (\alpha' + \gamma_1)$;
- $\mathrm{A}_h(T_2) = \pi - (\beta' + \gamma_2)$.

Using furthermore that $\gamma + \gamma_1 + \gamma_2 = 2\pi$ and re-injecting in $\mathrm{A}_h(T_3) = \mathrm{A}_h(T) + \mathrm{A}_h(T_1) + \mathrm{A}_h(T_2)$ we obtain

$$\mathrm{A}_h(T) = \pi - (\alpha + \beta + \gamma)$$

which was what we wanted to prove. ■

4.4.3 Pythagoras theorem

Theorem 14 (Hyperbolic Pythagoras). Let T be a right-angled hyperbolic triangle with vertices $p, q, r \in \mathbb{H}$ with sides of respective lengths a, b and c (where the side of length c is the side opposite to the right angle). We have

$$\cosh(c) = \cosh(a) \cdot \cosh(b).$$

Proof: Calculation done in the lectures using the formula for the hyperbolic distance. ■

4.5 Exercises

Exercise 56. *Recall that the hyperbolic length of a path* $\gamma : [a, b] \longrightarrow \mathbb{H}$ *is*

$$\mathrm{L}(\gamma) = \int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))} dt.$$

Show that for any Moebius map $T : z \mapsto \frac{az+b}{cz+d}$ *with* a, b, c *and* $d \in \mathbb{R}$ *and* $ad - bc > 0$ *we have*

$$\mathrm{L}(T \circ \gamma) = \mathrm{L}(\gamma).$$

Exercise 57. *We can define the hyperbolic distance between two points* $p, q \in \mathbb{H}$ *as*

$$d_h(p, q) = \inf_{\gamma : p \to q} \mathrm{L}(\gamma).$$

Using Exercise 56, find an alternative proof that maps of the form $T : z \mapsto \frac{az+b}{cz+d}$ *with* a, b, c *and* $d \in \mathbb{R}$ *and* $ad - bc > 0$ *are isometries of* $\mathbb{H}$.

Exercise 58. *Find the equation of the geodesic through the points* i *and* $1 + 4i$.

Exercise* 59. *Let* G_1 *and* G_2 *be two geodesics with respective points at infinity* (a_1, b_1) *and* (a_2, b_2). *Show that* G_1 *and* G_2 *intersect in* $\mathbb{H}$ *if and only if*

$$\mathrm{Cr}(a_1, b_1, a_2, b_2) < 0$$

Exercise* 60. *Let* G_1 *and* G_2 *be two geodesics with respective points at infinity* (a_1, b_1) *and* (a_2, b_2). *Show that* G_1 *and* G_2 *intersect orthogonally if and only if*

$$\mathrm{Cr}(a_1, b_1, a_2, b_2) = -1.$$

Exercise 61. *Let* p *and* q *in* $\mathbb{H}$. *Show that there exists* $f \in \mathrm{PGL}(2, \mathbb{R})$ *such that* $f(p) = q$.

Exercise 62. *Let* G_1 *and* G_2 *be two geodesics. Show that there exists* $f \in \mathrm{PGL}(2, \mathbb{R})$ *such that* $f(G_1) = G_2$.

Exercise 63. *Let* G_1 *and* G_2 *two geodesics lines through* i. *Show that there exits* θ *such that* $R_\theta := z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta + \cos\theta}$ *maps* G_1 *to* G_2.

Exercise* 64 (Homogeneity of $\mathbb{H}$)**.** *Using Exercises 61 and 63, show that for any two pairs* (p_1, G_1) *and* (p_2, G_2) *where* G_i *is a line through* p_i *for* $i = 1, 2$, *there exists* $f \in \mathrm{PGL}(2, \mathbb{R})$ *such that*

$$(f(p_1), f(G_1)) = (p_2, G_2).$$

Exercise 65 (Fixator of a point). *For $p \in \mathbb{H}$, define* $\mathrm{Fix}_p := \{T \in \mathrm{PGL}(2, \mathbb{R}) \mid T(p) = p\}$.

1. *Show that* Fix_p *is a subgroup of* $\mathrm{PGL}(2, \mathbb{R})$.
2. *Show that* Fix_p *is conjugate in* $\mathrm{PGL}(2, \mathbb{R})$ *to*

$$\{z \mapsto \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta} \mid \theta \in [0, \pi)\}$$

3. *Show that* Fix_p *is abstractly isomorphic, as a group, to* $(\mathbb{R}/\mathbb{Z}, +)$.

Exercise 66 (Stabiliser of a geodesic). *For $p \in \mathbb{H}$, define* $\mathrm{Stab}(G) := \{T \in \mathrm{PGL}(2, \mathbb{R}) \mid T(G) = G\}$.[1].

1. *Show that* $\mathrm{Stab}(G)$ *is a subgroup of* $\mathrm{PGL}(2, \mathbb{R})$.
2. *Show that* $\mathrm{Stab}(G)$ *is conjugate in* $\mathrm{PGL}(2, \mathbb{R})$ *to*

$$\{z \mapsto \lambda z \mid \lambda \in \mathbb{R}_{>0}\}$$

3. *Show that* $\mathrm{Stab}(G)$ *is abstractly isomorphic, as a group, to* $(\mathbb{R}, +)$.

Exercise 67 (Fixator of a point at infinity). *For $p \in \partial\mathbb{H} = \hat{\mathbb{R}}$, define* $\mathrm{Fix}_p := \{T \in \mathrm{PGL}(2, \mathbb{R}) \mid T(p) = p\}$.

1. *Show that* Fix_p *is a subgroup of* $\mathrm{PGL}(2, \mathbb{R})$.
2. *Show that* Fix_p *is conjugate in* $\mathrm{PGL}(2, \mathbb{R})$ *to*

$$\{z \mapsto \lambda z + \mu \mid \lambda \in \mathbb{R}_{>0}, \mu \in \mathbb{R}\}$$

3. *Show that* Fix_p *is abstractly isomorphic, as a group, to* $(\mathbb{R}/\mathbb{Z}, +)$.

[1] T need not fix G pointwise, just map G to itself

Appendix A

Algebra toolbox: groups, matrix groups and scalar products

A.1 Scalar products

A.1.1 Definition

Definition 29. A *scalar product* on an $\mathbb{R}$-vector space E of dimension d is a bilinear form $\langle \cdot, \cdot \rangle : E \times E \longrightarrow \mathbb{R}$ which is

- *symmetric* i.e. $\forall \vec{v}, w \in E$, $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$;
- *positive* i.e. $\forall \vec{v} \in E$, $\langle \vec{v}, \vec{v} \rangle \geq 0$;
- *definite* i.e. $\langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = 0$

The **canonical scalar product** on $\mathbb{R}^d$ is by definition

$$\langle \vec{v}, \vec{w} \rangle := \sum_{i=1}^{d} v_i w_i.$$

A.1.2 Matrix of a scalar product in a base and vector representation of a scalar product.

Scalar product defined by a matrix. Let $A = (a_{ij})$ be a symmetric matrix (*i.e.* $a_{ij} = a_{ji}$. Then

$$\begin{array}{ccc} \mathbb{R}^d \times \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ (X, Y) & \longmapsto & {}^t X A Y \end{array}$$

defines a symmetric bilinear form. It is not quite a scalar product as it could not be positive or definite (think of the case $A = 0$ for instance). This is easily checked:

- Y is a column vector, in other words a $d \times 1$ matrix;

- since A is a $d \times d$ matrix, AY is a $d \times 1$ matrix;
- ${}^tX$ is a line vector (as the transpose of the column vector X) i.e. a $1 \times d$ matrix;
- therefore ${}^tX \cdot (AY)$ is the product of a $1 \times d$ matrix by a $d \times 1$, it results in a 1×1 matrix, in other words a real number!

Matrix of a scalar product Let $B = (\vec{e_1}, \cdots, \vec{e_d})$ be a basis of a vector space E endowed with a scalar product $\langle \cdot, \cdot \rangle$. The *matrix of E in the basis B* is by definition the matrix

$$A := (\langle \vec{e_i}, \vec{e_j} \rangle)_{i,j}.$$

Since $\langle \cdot, \cdot \rangle$ is symmetric, so is the matrix A. If $\vec{v} = \sum_{i=1}^{d} v_i \vec{e_i}$ and $\vec{w} = \sum_{i=1}^{d} w_i \vec{e_i}$, we denote by V and W the vectors in $\mathbb{R}^d$ $(v_1, \cdots, v_d)$ and $(w_1, \cdots, w_d)$. By definition of A we have

$$\langle \vec{v}, \vec{w} \rangle =^t W \cdot A \cdot V =^t V \cdot A \cdot W$$

For instance, if $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^d$, its matrix in the canonical basis $B = (\vec{e_1}, \cdots, \vec{e_d})$ is I_d, as $\langle \vec{e_i}, \vec{e_j} \rangle = \delta_{ij}$.

Change of basis Let $B = (\vec{e_1}, \cdots, \vec{e_d})$ and $B' = (\vec{f_1}, \cdots, \vec{f_d})$ be two basis, A be the matrix of a scalar product $\langle \cdot, \cdot \rangle$ in the basis B and P be the matrix of the change of basis from B' to B. We have the following important fact

Proposition 24. The matrix of $\langle \cdot, \cdot \rangle$ in the basis B' is equal to

$${}^tP \cdot AP.$$

Proof: Let $\vec{x} \in \mathbb{R}^d$ be a vector, we write it in base B, $\vec{x} = \sum_i x_i \vec{e_i}$ and in base B' $\vec{x} = \sum_i x'_i \vec{f_i}$.

By definition of P we have

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = P \cdot \begin{pmatrix} x'_1 \\ \vdots \\ x'_d \end{pmatrix}.$$

By definition of the matrix of a scalar product in basis B

$$\langle \vec{x}, \vec{y} \langle =^t Y \cdot A \cdot X$$

which is also equal to

$$\langle \vec{x}, \vec{y} \langle =^t (P \cdot Y') \cdot A \cdot (PX')$$

with $X' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_d \end{pmatrix}$ and $Y = \begin{pmatrix} y'_1 \\ \vdots \\ y'_d \end{pmatrix}$. Therefore

$$\langle \vec{x}, \vec{y} \langle =^t Y' \cdot ({}^tPAP) \cdot X'$$

which shows that the matrix of $\langle \cdot, \cdot \rangle$ in the basis B' is equal to ${}^tPAP$. ∎

A.1.3 Every scalar product on $\mathbb{R}^d$ is equivalent to the canonical one.

Theorem 15. Let $\langle\cdot,\cdot\rangle$ a scalar product on $\mathbb{R}^d$.

1. Then there exists a basis in which the matrix of $\langle\cdot,\cdot\rangle$ is I_d.
2. Equivalently, if A is the matrix of $\langle\cdot,\cdot\rangle$ in the canonical basis, there exists $P \in \mathrm{GL}(d,\mathbb{R})$ such that
$$A =^t PP.$$
3. Equivalently, there exists an orthonormal basis for $\langle\cdot,\cdot\rangle$.

We do not give a proof of this theorem here.

A.2 Matrix groups

In this paragraph we discuss various matrix groups and some variations such as affine groups and projective groups. d is an integer and $\mathbb{K}$ is a field that is either $\mathbb{R}$ or $\mathbb{C}$ (even though most of what we are going to discuss holds for $\mathbb{Q}$ or even finite fields). Recall that we denote by $\mathrm{M}_d(\mathbb{K})$ the set of $d \times d$-matrices with entries in $\mathbb{K}$.

A.2.1 Linear and special linear groups.

Definition 30 (Linear group). The *linear group* of dimension d on the field $\mathbb{K}$ is the set

$$\mathrm{GL}(d,\mathbb{K}) = \{M \in \mathrm{M}_d(\mathbb{K}) \mid \det A \neq 0\}.$$

One can easily check (do it as an exercise!) that $\mathrm{GL}(d,\mathbb{K})$ with the multiplication of matrices forms a group.

Definition 31 (Special linear group). The *special linear group* of dimension d on the field $\mathbb{K}$ is the set

$$\mathrm{SL}(d,\mathbb{K}) = \{M \in \mathrm{M}_d(\mathbb{K}) \mid \det A = 1\}.$$

Since $\mathrm{SL}(d,\mathbb{K}) = \mathrm{Ker}(\det)$ where det is seen as a group homomorphism $\mathrm{GL}(d,\mathbb{K}) \longrightarrow \mathbb{K}^*$, it is a subgroup of $\mathrm{GL}(d,\mathbb{K})$.

A.2.2 Projective groups

We now introduce the projective group. The group $\mathrm{GL}(d,\mathbb{K})$ contains a subgroup isomorphic to $\mathbb{K}^*$, that is the group

$$\{k \times \mathrm{I}_d \mid k \in \mathbb{K}^*\}.$$

We will denote this group $\mathbb{K}^*$, even if this is somewhat improper notation. Note that $\mathbb{K}^*$ is a normal subgroup of $\mathrm{GL}(d, \mathbb{K})$.

Definition 32 (Projective group). The *projective group* of dimension d on the field $\mathbb{K}$ is the set

$$\mathrm{PGL}(d, \mathbb{K}) = \mathrm{GL}(d, \mathbb{K})/\mathbb{K}^*$$

Proposition 15. *The multiplication of* $\mathrm{GL}(d, \mathbb{K})$ *passes to the quotient* $\mathrm{PGL}(d, \mathbb{K}) = \mathrm{GL}(d, \mathbb{K})/\mathbb{K}^*$. *The set* $\mathrm{PGL}(d, \mathbb{K}) = \mathrm{GL}(d, \mathbb{K})/\mathbb{K}^*$ *together with this composition law is a group.*

Proof: Exercise. ■

A.2.3 Generation of $\mathrm{SL}(2, \mathbb{K})$

Lemma 16. *The group* $\mathrm{SL}(2, \mathbb{K})$ *is generated by matrices of the form* $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $z \in \mathbb{K}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda \in \mathbb{K}^*$ *and* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.