A SHORT NOTE ABOUT THE RENORMALISATION OF GENERALISED IETS.

In this note we discuss some geometric properties of renormalisation for generalised interval exchange transformations. **All transformations will be assumed to be of class $\mathcal{C}^\infty$.** The renormalisation operator is Rauzy induction.

Question 1. *Let T be a minimal GIET with mean non-linearity zero and reasonable rotation number. Is it true that renormalisations of T converge to affine IETs?*

One of the goals of this note is to convince the reader that this question is linked to the ergodic properties of T with respect to Lebesgue measure.

We denote by $\mathcal{R}$ the Rauzy-Veech induction acting upon the space of generalised interval exchange transformations. For the remainder of this text, T is a GIET with infinite complete Rauzy path. Let n be its number of continuity intervals. Recall the following well-known properties of renormalisation:

- if T is minimal, renormalisation converges towards projective IETs
- there is an obstruction to convergence to affine which is the mean non-linearity

  $$\int_0^1 \eta_T d\mathrm{Leb}$$

  where $\eta_f = D(\log Df)$. This number has to vanish for convergence towards affine IETs to happen;
- in the event of convergence to affine, there is a cohomological obstruction to converge to linear IETs.

In this note we investigate a possible additional obstruction which has to do with Lebesgue ergodicity.

Remark 1. *To talk about Lebesgue ergodicity for a diffeomorphism, one does not equire that the Lebesgue measure is invariant, but **quasi-invariant** which is always the case with piecewise diffeomorphisms. It is in this context that we place ourselves.*

1. Rauzy-Veech induction

We review very briefly dynamical partitions associated with Rauzy induction (which we do not redefine, see [Yoc09]).

1.1. **Dynamical partitions.** Let $I_1^0, \cdots, I_n^0$ be the continuity intervals of T. The image by Rauzy induction of T is a first return map on a strict subinterval of $[0,1]$. Hence $\mathcal{R}^k T$ is a first return map on an interval $[0, a_k] \subset [0,1]$. Let $I_1^k, \cdots, I_n^k$ be the continuity intervals of $\mathcal{R}^k T$. We denote by $\mathcal{P}_1^k, \cdots, \mathcal{P}_n^k$ the dynamical partition at stage k of T where

$$\mathcal{P}_i^k = \bigcup_{i=0}^{l_i^k} T^j(I_i^k)$$

by definition and l_i^k is the first return time of I_i^k to $[0, a_k]$. The $\mathcal{P}_i$ form a partition of $[0,1]$. We also define the size of a partition as being

$$M_k = \sup_{i \leq n,\ j \leq l_i^k} |T^j(I_i^k)|.$$

We have the following proposition

Proposition 1. *T is minimal if and only if M_k tends to zero as k tends to infinity.*

The $\mathcal{P}_i$ form a partition of $[0,1]$.

1.2. **Partitional measures.** We now define

$$\mu_i^k = \mathrm{Leb}_{|\mathcal{P}_i^k}.$$

Lemma 1. *Assume T is ergodic with respect to Lebesgue measure. Then for all i,*

$$\frac{\mu_i^k}{|\mu_i^k|} \text{ converges to } \mathrm{Leb}.$$

Proof. First, notice that all the μ_i^k are absolutely continuous with respect to Lebesgue. Second, notice that

$$T_* \frac{\mu_i^k}{|\mu_i^k|} - DT. \frac{\mu_i^k}{|\mu_i^k|}$$

is very small because $\mathcal{P}_i^k$ is almost invariant by the action of f. It means that any weak-limit point μ_∞ of the sequence $\frac{\mu_i^k}{|\mu_i^k|}$ has to satisfy

$$T_* \mu_\infty - DT.\mu_\infty = 0.$$

The measure μ_∞ also has to be absolutely continuous with respect to Lebesgue. This two last points, together with Lebesgue ergodicity of T imply that μ_∞ is the Lebesgue measure. Indeed because μ_∞ is absolutely continuous with respect to Lebesgue there exist a positive measurable function g such that

$$\mu_\infty = g \cdot \mathrm{Leb}$$

Because both μ_∞ and Leb satisfy the equation

$$T_*\mu - DT.\mu = 0$$

g has to be invariant by the action of T. Because T is Lebesgue ergodic, g is constant almost everywhere hence μ_∞ is the Lebesgue measure.

□

2. Convergence of renormalisation

2.1. **Convergence to projective.** The following result is well known

Proposition 2. *Let T be a minimal interval exchange transformation. Then there exists $C > 0$ such that*

$$d_{\mathcal{C}^2}(\mathcal{R}^k T, \textit{Projective}) \leq C M_k$$

.

2.2. **Conditional convergence to affine.** We prove here that if T is a Lebesgue-ergodic, projective IET then its renormalisations converge to affine.

Proposition 3. *Let T be a minimal, Lebesgue-ergodic, projective interval exchange transformation with mean zero non-linearity. Then*

$$d_{\mathcal{C}^2}(\mathcal{R}^k T, \textit{Affine})$$

tends to zero as k tends to infinity.

Proof. Recall that $\mathcal{R}^k T$ is the first return map of T on a certain subinterval and that $I_1^k, \cdots, I_n^k$ are its intervals of continuity.

By the chain rule, we know that

$$\int_{I_i^k} \eta(\mathcal{R}^k T) \mathrm{d}Leb = \int_{\mathcal{P}_i^k} \eta(T) \mathrm{d}Leb$$

By definition of μ_i^k

$$\int_{I_i^k} \eta(\mathcal{R}^k T) \mathrm{d}Leb = \int_{[0,1]} \eta(T) \mathrm{d}\mu_i^k$$

and since $\frac{\mu_i^k}{|\mu_i^k|}$ converges to Lebesgue and that $\int \eta(T) = 0$, we get that

$$\int_{I_i^k} \eta(\mathcal{R}^k T) \mathrm{d}Leb$$

converges to zero for all i. The proposition is easily deduced from this fact.

□

2.3. **Comments.**

On the speed of convergence. Unfortunately this type of reasoning only gives abstract convergence with no quantitative estimate whatsoever. It is expected that the speed of convergence is somewhat controlled by the size of the dynamical partition. We wonder if the argument in the proof of Lemma 1 can be made quantitative.

On Lebesgue ergodicity. It seems apparent that Lebesgue ergodicity and renormalisation talk to each other. However it is not so clear how deep this link is. It could be that it is just an artefact in the proof. However, all proofs of convergence in the case of circle diffeomorphisms use statements somewhat equivalent to ergodicity.

3. Open questions and further comments

We close this short note with a few questions.

Question 2. *Is any minimal, uniquely ergodic affine interval exchange tranformation Lebesgue ergodic?*

Question 3. *Is there an example of minimal, uniquely ergodic generalised interval exchange tranformation which is not Lebesgue ergodic?*

Question 4. *Let T be a minimal GIET. Can one get a speed of convergence for M_k, maybe in terms of reasonable acceleration of the induction? (Yoccoz acceleration for instance)*

Question 5. *Is there a minimal generalised interval exchange transformation T for which the sequence $\mathcal{R}^n T$ is not shadowed by that of a projective IET?*

Question 6. *How does the sequence of weights $(|\mu_1^k|, \cdots, |\mu_1^k|)$ of the "partitional" measures behave?*

Question 7. *Are there examples of minimal uniquely ergodic GIETs whith mean zero non-linearity for which renormalisation does not converge to affine?*

On Lebesgue ergodicity for minimal diffeomorphisms. It seems to us that the question of Lebesgue ergodicity is interesting in its own right. The type of questions relative to this measure always resonate with physical interpretation (is one is ready to believe that the Lebesgue measure is actually that one observes in the real world).

It is a standard result that $\mathcal{C}^2$-circle diffeomorphisms are always Lebesgue ergodic (even when the invariant measure is singular), see [KH95]. It is no more the case in dimension 2 where Yoccoz (see [Yoc80]) has given an example of minimal diffeomorphism which is not Lebesgue ergodic. Finally, note there are know example of minimal (linear) interval exchange transformation for which Lebsegue measure is not ergodic (see [Kea77]). This is why we often ask for T to be uniquely ergodic, to avoid Lebesgue non-ergodicity coming from pure combinatorics.

References

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