Math 355: Intro to Analysis
Practice Midterm Exam #2

- Attempt problems 1-6.
- **Justify all your answers**. (If you are unsure whether or not something requires further justification, you are welcome to ask me.)
- Please clearly identify any theorems or previous results that you use.
- Please write clearly and legibly, and cross out or erase anything that you do not want graded.
- You may use the course textbook, your class notes, and old homework and exams. You may also use a calculator or graphing software to graph functions.
- No other textbooks, websites, or outside help may be used on this exam.
- All discussion about this exam is strictly prohibited (including conversations about how easy/hard a question is, and how much progress you have made so far).

1. (20 points) Determine whether the following are **true** or **false**. Please carefully justify all your answers.
   (a) The arbitrary union of compact sets is compact.
   (b) Every nonempty open set contains a rational number.
   (c) If $f : A \to \mathbb{R}$ is a continuous function and $F \subset A$ is a closed set then $f(F)$ is an closed set.
   (d) If $f : A \to \mathbb{R}$ is a continuous function and $B \subset A$ is a bounded set then $f(B)$ is a bounded set.
2. (15 points) Let (f_n) be a sequence of functions $A \to \mathbb{R}$ that are each uniformly continuous and converge uniformly to a limit f on A. Is f necessarily uniformly continuous?
3. (15 points) Given a differentiable function $f : A \to \mathbb{R}$, we will say that f is *uniformly differentiable* on A if given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon$$

whenever $0 < |x - y| < \delta$.

(a) Is $f(x) = x^2$ uniformly differentiable on $\mathbb{R}$? How about $g(x) = x^3$?

(b) Show that if a function is uniformly differentiable on an interval A, then the derivative must be continuous on A.

4. (20 points) Give an example of:

   (a) continuous functions f and g which are not differentiable at zero but such that fg is differentiable at zero.

   (b) a collection of open sets $\{U_1, U_2, \ldots\}$ such that $\cap_{n=1}^{\infty} U_i$ is a closed, nonempty, proper subset of $\mathbb{R}$.

   (c) two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both of which diverge but for which $\sum_{n=1}^{\infty} a_n b_n$ converges.

   (d) A continuous function defined on an open interval with range equal to a closed interval.

5. (15 points) Use the Mean Value Theorem to prove that for all real numbers $x < y$, we have $|\cos(x) - \cos(y)| \leq |x - y|$. Prove that $\cos(x)$ is uniformly continuous on $\mathbb{R}$.

6. (15 points) Let $f_n(x) = \frac{nx}{1+nx^2}$.

   (a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.

   (b) Is the convergence uniform on $(0, \infty)$?

   (c) Is the convergence uniform on $(0, 1)$?

   (d) Is the convergence uniform on $(1, \infty)$?