FREE ELECTRON MODEL

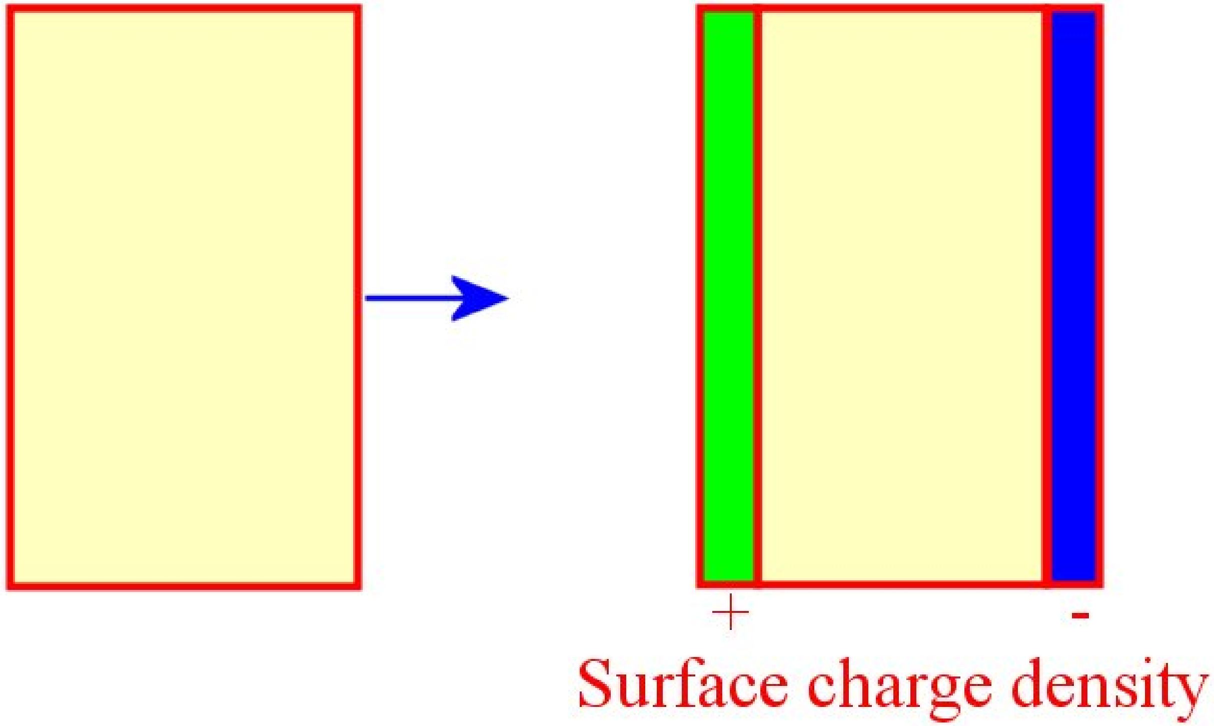
Lecture 17

A.H. Harker
Physics and Astronomy
UCL

Magnetic Effects

6.7 Plasma Oscillations

The picture of a free electron gas and a positive charge background offers the possibility of *plasma oscillations* – a collective motion of all the electrons relative to the background.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Surface charge density

If electron gas, n electrons per volume, moves a distance x relative to the positive background this gives a surface charge density

$$\sigma = -enx$$

on the positive x side. But this gives an electric field

$$\mathcal{E} = -\frac{\sigma}{\epsilon_0},$$

which tries to restore the electrons to their equilibrium position by exerting a force

$$\mathcal{F} = -e\mathcal{E} = -\frac{ne^2}{\epsilon_0}x$$

on each electron. So

$$m\ddot{x} = -\frac{ne^2}{\epsilon_0}x,$$

which is a simple harmonic oscillator with angular frequency ω_{P}

$$\omega_{\mathrm{P}}^2 = \frac{ne^2}{\epsilon_0 m}.$$

For example, if $n = 6 \times 10^{28}\ \mathrm{m}^{-3}$**,**

$$\omega_{\mathrm{P}} = \sqrt{\frac{ne^2}{\epsilon_0 m}} = \sqrt{\frac{6 \times 10^{28} \times (1.6 \times 10^{-19})^2}{8.854 \times 10^{-12} \times 9.11 \times 10^{-31}}} = 1.4 \times 10^{16}\ \mathrm{rad\ s}^{-1}.$$

This corresponds to an energy

$$\hbar\omega_{\mathrm{P}} = 8.9\ \mathrm{eV}.$$

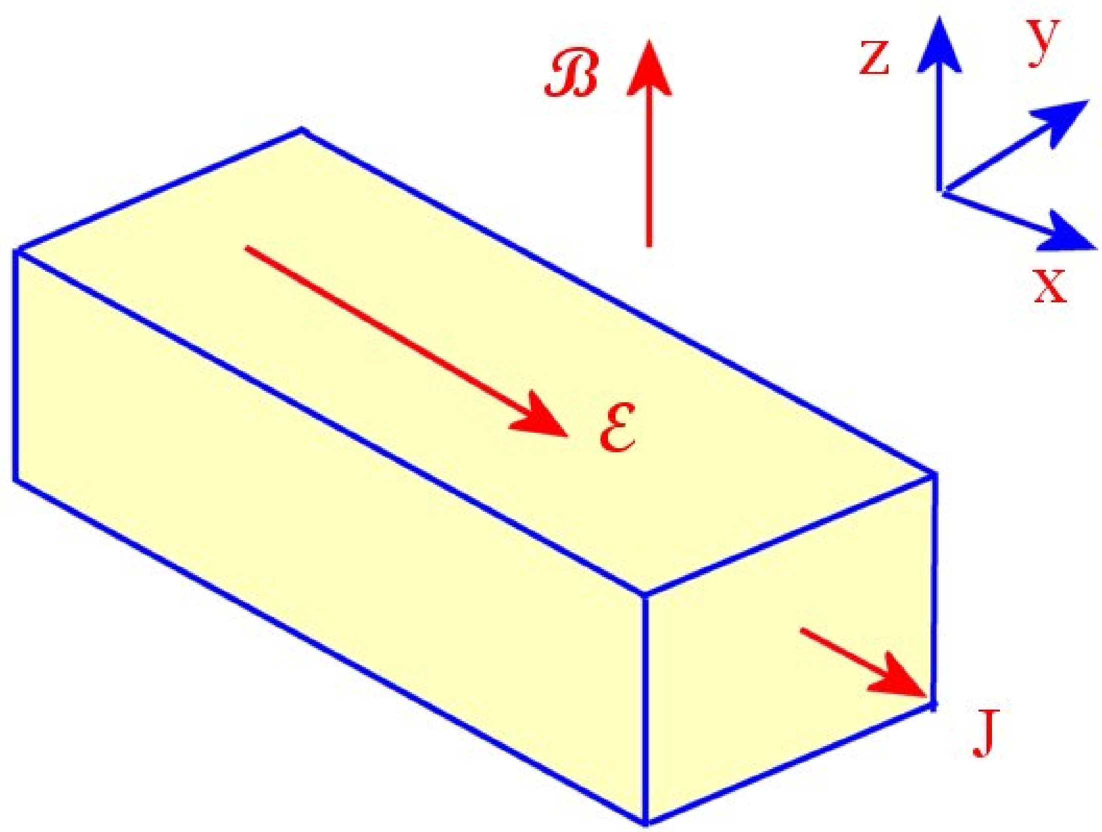
If high energy (1 to 10 keV) electrons are fired through a metal film, they can lose energy by exciting plasma oscillations, or *plasmons*.

Volume plasmon energies, eV		
Metal	**Measured**	**Calculated**
Li	**7.12**	**8.02**
Na	**5.71**	**5.95**
K	**3.72**	**4.29**
Mg	**10.6**	**10.9**
Al	**15.3**	**15.8**

Another success for free electron theory.

6.8 The Hall Effect

In a Hall experiment a magnetic field applied perpendicular to an electric current flowing along a bar.



We need to extend our previous equation by including the Lorentz force $q\mathbf{v} \times \mathcal{B}$.

Signs always cause problems in the Hall effect: avoid some confusion by writing q for the charge on the particles carrying the current – q includes the sign. The new transport equation is

$$m\left(\frac{\mathrm{d}\mathbf{v}_{\mathrm{d}}}{\mathrm{d}t}+\frac{\mathbf{v}_{\mathrm{d}}}{\tau}\right)=q(\mathcal{E}+\mathbf{v}_{\mathrm{d}}\times\mathcal{B}).$$

Asssume that $\mathcal{B}=(0,0,\mathcal{B}_z)$ **and** $\mathcal{E}=(\mathcal{E}_x,\mathcal{E}_y,\mathcal{E}_z)$ **so**

$$\begin{aligned}
m\frac{\mathrm{d}v_{\mathrm{d}x}}{\mathrm{d}t}+m\frac{v_{\mathrm{d}x}}{\tau} &= q\mathcal{E}_x+qv_{\mathrm{d}y}\mathcal{B}_z,\\
m\frac{\mathrm{d}v_{\mathrm{d}y}}{\mathrm{d}t}+m\frac{v_{\mathrm{d}y}}{\tau} &= q\mathcal{E}_y-qv_{\mathrm{d}x}\mathcal{B}_z,\\
m\frac{\mathrm{d}v_{\mathrm{d}z}}{\mathrm{d}t}+m\frac{v_{\mathrm{d}z}}{\tau} &= q\mathcal{E}_z.
\end{aligned}$$

Now we know that current can only flow in the x direction, so $v_{\mathrm{d}y} = v_{\mathrm{d}z} = 0$, and so in a steady state

$$\begin{aligned} m\frac{v_{\mathrm{d}x}}{\tau} &= q\mathcal{E}_x, \\ 0 &= q\mathcal{E}_y - qv_{\mathrm{d}x}\mathcal{B}_z, \\ 0 &= q\mathcal{E}_z. \end{aligned}$$

$$
\begin{aligned}
m\frac{v_{\mathrm{d}x}}{\tau} &= q\mathcal{E}_x, \\
0 &= q\mathcal{E}_y - qv_{\mathrm{d}x}\mathcal{B}_z, \\
0 &= q\mathcal{E}_z.
\end{aligned}
$$

The first equation is one we have seen before:

$$v_{\mathrm{d}x} = \frac{q\tau}{m}\mathcal{E}_x,$$

giving the current along the bar.

The third equation states that there is no electric field in the z **direction.**

The second equation states that an electric field is set up in the y **direction:**

$$\mathcal{E}_y = v_{\mathrm{d}x}\mathcal{B}_z.$$

Physically what happens is that the charges are accelerated in the y **direction by the magnetic field, and pile up on the edges of the bar until they produce enough of an electric field to oppose the effect of the magnetic field.**

We know that the current density J_x in the x direction is

$$J_x = nqv_{\mathrm{d}x},$$

so

$$\mathcal{E}_y = \frac{J_x \mathcal{B}_z}{nq},$$

and we define the Hall coefficient as

$$R_{\mathrm{H}} = \frac{\mathcal{E}_y}{J_x \mathcal{B}_z}.$$

For a free electron metal with n electrons per volume, then, R_{H} is negative,

$$R_{\mathrm{H}} = -\frac{1}{ne}.$$

Note that measuring Hall effects in metals is *difficult*: even with high current density ($10^6\ \mathrm{Am}^{-2}$ and magnetic fields of order $1\ \mathrm{T}$ we have to measure fields

$$\mathcal{E}_y = \frac{10^6 \times 1}{6 \times 10^{28} \times 1.6 \times 10^{-19}} = 0.0001\ \mathrm{V\ m}^{-1},$$

or a potential difference of less than $1\ \mu\mathrm{V}$ on a typically-sized sample.

Metal	**Valence**	$R_{\mathrm{H}}^{\mathrm{theor}}/R_{\mathrm{H}}^{\mathrm{exp}}$
Li	**1**	**0.8**
Na	**1**	**1.2**
K	**1**	**1.1**
Rb	**1**	**1.0**
Cs	**1**	**0.9**

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Cs	1	0.9
Cu	1	1.5
Ag	1	1.3
Au	1	1.5
Be	2	-0.2
Cd	2	-1.2
Zn	3	-0.8
Al	3	-0.3

Alkali metals OK, noble metals numerically incorrect, higher-valent metals *wrong sign*. Major problem for free-electron theory. In addition, R_{H} depends on $\mathcal{B}$ and T.

6.9 Free electron approximation – final comments

We have still not explained how we can justify the assumption that electrons, charged particles, do not interact with one another. There are two effects – electrostatic screening and the exclusion principle.

6.9.1 Screening

If the electrons are free to move, they arrange themselves so as to make the metal locally neutral – but if they try to pack together more densely this will increase their energy because E_{F}, the energy relative to the local potential, increases with $n = N_e/V$. As a result, the electrostatic potential round a point charge q in a free electron gas is not

$$\mathcal{V}_0(r) = \frac{q}{4\pi\epsilon_0 r},$$

but

$$\mathcal{V}(r) = \frac{q e^{-r/\lambda}}{4\pi\epsilon_0 r},$$

a *screened Coulomb potential*, with

$$\lambda = \sqrt{\frac{2\epsilon_0 E_{\mathrm{F}}}{3e^2 n}} \approx 6 \times 10^{-11}\ \mathrm{m}$$

for our usual set of parameters, so that electric fields inside a metal are screened out within a few interatomic spacings.

6.9.2 Electron-electron scattering

At absolute zero, scattering *cannot* occur, because of the exclusion principle:
The two electrons are initially both in occupied states inside the Fermi surface.
To conserve energy and momentum, either both final states lie inside the Fermi surface – but those states are all occupied – or one lies outside – but then the other lies inside.
Scattering is forbidden at $T = 0$.
At finite T there is a layer of partly occupied states near E_{F}, amounting to a fraction about $k_{\mathrm{B}}T/E_{\mathrm{F}}$ of the electrons, giving weak scattering with probability $\propto T^2$.
See contribution to electrical resistivity $\propto T^2$ in very pure metals at very low T.

6.9.3 Binding energy of metals

The terms in the energy are:

- **Electronic kinetic energy (reduced by allowing them to be delocalised)**
- **Attraction of electrons to ion cores (less than in free atoms as electrons are further from nuclei)**
- **Mutual repulsion of ion cores (screened by the free electron gas)**
- **Electron-electron repulsion (reduced by spreading out electrons)**
- **Quantum mechanical exchange potential between electrons**
- **Correlation energy (beyond single-electron wave-functions)**

Balance of effects – typically a few eV per atom.