Spin–orbit forces in P-wave baryons from a three-quark flux-tube potential

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Abstract. Using a Foldy–Wouthuysen transformation, the spin–orbit interaction is derived for a three-quark system confined by a realistic three-body flux-tube potential. The result is used together with harmonic oscillator wavefunctions to examine spin–orbit splittings in P-wave baryons. It is shown that the splittings are similar to those which arise from an approximate confining potential involving two-body potentials.

1. Introduction

The quark model with interactions suggested by quantum chromodynamics has achieved some remarkable successes in attempting to account for the mass spectrum and decay characteristics of hadrons [1]. An important part of this effort has been the study of the spin-dependent splittings in the mass spectrum. A significant step forward [2] was the consideration of a chromomagnetic hyperfine interaction term in the quark Hamiltonian, arising from single gluon exchange between pairs of quarks. The model provides spin–spin and tensor forces which have helped to explain the spectrum of a number of low-lying mesons and baryons [3]. What is more, the results are parametrised solely by the quark mass and the strong coupling constant. However, for hadronic states which contain quarks with orbital angular momentum, spin–orbit forces must be calculated, and this is where the simplest quark models fail. The one gluon exchange model implies a spin–orbit force for the baryon system which is about an order of magnitude too large, when the parameters are fixed by the spin–spin splittings [3, 4].

A partial solution to this problem has long been recognised [3]. The confining potential between quarks is thought to be scalar in character. If so, then it provides a spin–orbit force, due to Thomas precession, which is opposite in sign to that due to one gluon exchange [5]. There is support for this idea from non-perturbative studies of spin–orbit potentials in quarkonia [6, 7]. This explanation for the small spin–orbit splittings has not met with success in baryons, however, where detailed calculation with trial confining potentials has shown that the cancellation applies in some, but not all, of the states [3].

A certain amount of progress has been made by adding complexity, through adjustable parameters, to the original quark potential model. A 'relativised' quark Hamiltonian [8] can successfully describe a large amount of spectroscopic and decay

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data in both mesons and baryons. The spin–orbit effects can be reduced in size relative to the spin–spin contributions, essentially by smearing and adding momentum dependence to the potentials.

Other approaches have been used which have aimed to reduce the spin–orbit forces relative to the spin–spin contributions. Amongst these there are considerations of pion exchange between quarks [9, 10], which introduces additional spin–spin and tensor splittings. Some models consider deformed potentials for the excited states [10], and others take the view that non-local effects in the confining potential suppress the spin–orbit forces [11].

This paper seeks to address a question which has been ignored even within the recent complicated models. The confining potential in baryons, according to flux-tube [12] ideas, is a three-body interaction [13], but up to now, has been treated as a sum of effective two-body potentials for the purposes of calculating the Thomas effect [3, 8]. The treatment is reasonably well justified when considering the non-relativistic spectrum (where the relative error is less then 15% [13]) but for the Thomas term, derivatives of the non-relativistic potentials are required and it remains to be demonstrated that such an approximation is adequate. This question is pursued here.

The following section describes the three-quark flux-tube potential and gives a derivation of the associated Thomas term using a Foldy–Wouthuysen transformation [14] applied to a three-particle system. This generalises the two-particle case considered by Chraplyvy [15]. Section 3 contains calculations of the spin–orbit splittings in P-wave baryons due to the derived Thomas and one gluon exchange interactions. Harmonic oscillator wavefunctions with equal mass quarks are used. These results are compared with the two-body approximation and then our conclusions are given in §4.

2. The three-body potential and Thomas term

According to the flux-tube model, the confining mechanism in hadrons is the binding together of the quarks by gluonic flux constrained within narrow tubes. The tubes can only be broken by the creation of a quark–antiquark pair to terminate the newly formed ends. The rules of SU(3) symmetry dictate that each quark is the source of a flux-tube, but also that three tubes may intersect at a point [16].

The energy of a flux-tube is proportional to its length. The confining potential in baryons may therefore be found by minimising the total interconnecting length of flux-tubes, according to the above rules, for a particular quark configuration. The gluonic energy is assumed to follow the motion of the quarks adiabatically, such that it is a function solely of their coordinates. The (scalar) confining potential for the three-quark system is then given by

$$V_{\mathrm{i}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = K(r_{12} + r_{23}) \tag{1}$$

$$V_{\mathrm{ii}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = K(r_{12} + r_{13}) \tag{2}$$

$$V_{\mathrm{iii}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = K(r_{13} + r_{23}) \tag{3}$$

$$V_{\mathrm{iv}}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = K(r_{14} + r_{24} + r_{34}) \tag{4}$$

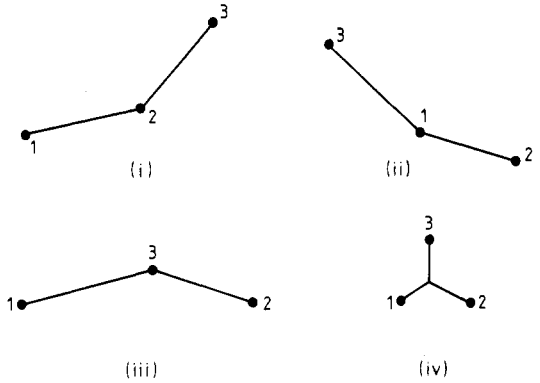

Figure 1. Possible flux-tube arrangements in a three-quark system.

for the flux-tube configurations of the types (i)–(iv) shown in figure 1. The quark coordinates are r_{1-3}, r_{ij} is the distance between quarks i and j, and r_{i4} is the distance from quark i to the junction of the three flux-tubes in configuration (iv). K is the string tension. The flux-tubes entering the three-pronged junction in configuration (iv) are arranged at 120° to one another. In configurations (i)–(iii) the quark coordinates are such that the flux-tube junction lies at one of the quark positions.

The flux-tube potential will be referred to as $V(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ with the different forms understood. It is convenient to use the centre-of-mass coordinate $\boldsymbol{R}$ (for equal mass quarks) and the internal coordinates $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ defined by

$$\begin{aligned} \boldsymbol{R} &= \frac{1}{\sqrt{3}}(\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3) \\ \boldsymbol{\lambda} &= \frac{1}{\sqrt{6}}(\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3) \\ \boldsymbol{\rho} &= \frac{1}{\sqrt{2}}(\boldsymbol{r}_1 - \boldsymbol{r}_2). \end{aligned} \tag{5}$$

Figure 2 shows the meaning of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$. The versions of V can be written [8, 13]

$$V_{\mathrm{i}} = K\left[\sqrt{2}\rho + \left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 - \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2}\right] \tag{6}$$

$$V_{\mathrm{ii}} = K\left[\sqrt{2}\rho + \left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 + \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2}\right] \tag{7}$$

$$V_{\mathrm{iii}} = K\left[\left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 - \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2} + \left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 + \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2}\right] \tag{8}$$

$$V_{\mathrm{iv}} = K\{\tfrac{3}{2}(\rho^2 + \lambda^2) + 3[\lambda^2\rho^2 - (\boldsymbol{\lambda}\cdot\boldsymbol{\rho})^2]^{1/2}\}^{1/2} \tag{9}$$

where $\rho = |\boldsymbol{\rho}|$ and $\lambda = |\boldsymbol{\lambda}|$. The boundaries separating the various flux-tube configurations in (λ, ρ) space can be found easily. V_{i} is applicable for

$$\lambda > \frac{\rho}{2\sin(\pi/3 - \alpha)} \qquad \alpha < \pi/3 \tag{10}$$

where α is the angle between $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$. Similarly, V_{ii} applies for

$$\lambda > \frac{\rho}{2\sin(\alpha - 2\pi/3)} \qquad \alpha > 2\pi/3 \tag{11}$$

and V_{iii} for

$$\lambda < \frac{\rho}{3}\left[5 - 2\cos^2\alpha - 2(4 - 5\cos^2\alpha + \cos^4\alpha)^{1/2}\right]^{1/2}. \tag{12}$$

Otherwise, V_{iv} is the correct form of the potential.

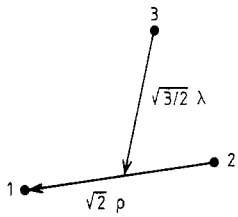

Figure 2. Internal coordinates λ and ρ.

It is convenient at this point to note the form of the two-body flux-tube potential which has been used to approximate the above three-body potential. It is

$$V_{2\mathrm{b}} = fK(r_{12} + r_{13} + r_{23}) \tag{13}$$

which introduces an error characterised by the difference potential:

$$V_{3\mathrm{b}} = V - fK(r_{12} + r_{13} + r_{23}) \quad . \tag{14}$$

A value of $f = \frac{1}{2}$ was used in [13] and the effect of the term $V_{3\mathrm{b}}$ upon the non-relativistic spectrum ignored (in addition, it seems that V_{iv} was used for all quark configurations). In [8], the value of f was chosen to be equal to 0.5493 in order to minimise the expectation value of $V_{3\mathrm{b}}$ within the harmonic oscillator representation of the ground state baryon. $V_{2\mathrm{b}}$ was used to derive spin–orbit forces due to the Thomas effect, whilst $V_{3\mathrm{b}}$ appears to have been neglected in this connection. As stated in the introduction, it is the effect of this neglect which is examined here.

Now we turn to the derivation of the Thomas term due to the potential V. A very useful tool for generating all of the relativistic corrections to a non-relativistic potential, in an expansion in powers of inverse mass, is the Foldy–Wouthuysen (FW) transformation. It is used here as an alternative to the methods based upon two-body scattering theory considered elsewhere [8]. Indeed, we use it expressly in order to consider three-body potentials.

The FW transformation is a well known procedure for partially block-diagonalising a Hamiltonian acting on four component Dirac spinors. The result is a Hamiltonian acting on two component Pauli spinors, obtained as an expansion in inverse mass. The first term in this expansion is the non-relativistic Hamiltonian, and the higher terms the relativistic corrections, including spin dependent pieces.

Complete block-diagonalisation is, however, only possible for the free-particle Dirac Hamiltonian:

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m \tag{15}$$

where m is the quark mass, $\boldsymbol{p}$ its momentum, and α_i and β are the 4×4 Dirac matrices which satisfy the usual anticommutation relations

$$\{\alpha_i, \beta\} = 0 \qquad \{\alpha_i, \alpha_j\} = 2\delta_{ij} \tag{16}$$

and are represented by

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{17}$$

using the Pauli matrices σ_i. It is easily shown that the transformed Hamiltonian

$$H' = \mathrm{e}^{S} H \mathrm{e}^{-S} \tag{18}$$

with

$$S = \frac{1}{2p} \tan^{-1}(p/m) \beta \boldsymbol{\alpha} \cdot \boldsymbol{p} \tag{19}$$

is block diagonalised as

$$H' = \beta (p^2 + m^2)^{1/2} \tag{20}$$

where $p = |\boldsymbol{p}|$ [17] . H' is thus transformed into two 2×2 blocks describing the free quark and antiquark. The quark Hamiltonian is obtained by replacing β by unity.

In the presence of interactions, however, the FW transformation can, in general, only elevate the non-block-diagonal contributions to the Hamiltonian to higher order in $(1/m)$. These contributions are referred to as 'odd', and generically contain α matrices. The desired terms in the Hamiltonian are known as 'even' and are proportional to β or the unit matrix. For example, consider the Hamiltonian

$$H_{\mathrm{I}} = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m + U \tag{21}$$

where U is a potential term. The odd term in order m^0 can be removed in exchange for an odd term proportional to m^{-1} with the choice

$$S = \frac{1}{2m} \beta \boldsymbol{\alpha} \cdot \boldsymbol{p} \tag{22}$$

giving

$$H_{\mathrm{I}}' = \mathrm{e}^{S} H_{\mathrm{I}} \mathrm{e}^{-S} = H_{\mathrm{I}} + [S, H_{\mathrm{I}}] + \tfrac{1}{2} [S, [S, H_{\mathrm{I}}]] + \ldots \tag{23}$$

in which an odd term of order m^0 in $[S, H_{\mathrm{I}}]$ cancels that in H_{I}:

$$[S, \beta m] = \frac{1}{2m} [\beta \boldsymbol{\alpha} \cdot \boldsymbol{p}, \beta m] = -\boldsymbol{\alpha} \cdot \boldsymbol{p}. \tag{24}$$

In exchange, there is the odd term

$$[S, U] = \frac{\beta}{2m} [\boldsymbol{\alpha} \cdot \boldsymbol{p}, U] \tag{25}$$

plus a number of further terms in order m^{-1} and above, both even and odd, coming from the $\frac{1}{2}[S,[S,H_I]]$ term. Further transformations using suitably chosen operators S can banish the odd terms to ever higher orders in $(1/m)$, leaving the non-relativistic Hamiltonian plus relativistic corrections as even terms. The desired spin-dependent relativistic corrections may be obtained by careful identification of the even terms which arise out of each FW transformation.

Now consider a quark–antiquark system with a scalar interaction alone between the quarks. In order to obtain the usual Thomas term due to a scalar confining potential V_c in quarkonium [5] the relativistic Hamiltonian is written as

$$H_{q\bar{q}} = \boldsymbol{\alpha}_1 \cdot \boldsymbol{p}_1 + \boldsymbol{\alpha}_2 \cdot \boldsymbol{p}_2 + \beta_1 m - \beta_2 m - \beta_1 \beta_2 V_c \tag{26}$$

where the labels 1 and 2 denote the quark and antiquark respectively. The Thomas term from the $\frac{1}{2}[S,[S,H_{q\bar{q}}]]$ part of the transformed Hamiltonian, now with

$$S = \frac{1}{2m}(\beta_1 \boldsymbol{\alpha}_1 \cdot \boldsymbol{p}_1 - \beta_2 \boldsymbol{\alpha}_2 \cdot \boldsymbol{p}_2) \tag{27}$$

is found, as required, to be

$$\frac{-1}{4m^2 r}\frac{\mathrm{d}V_c}{\mathrm{d}r}\boldsymbol{s}\cdot\boldsymbol{L} \tag{28}$$

where $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$, $\boldsymbol{s}$ is the total spin and $\boldsymbol{L}$ the total angular momentum. The non-relativistic form of the scalar potential is found by putting $\beta_1 = 1$, $\beta_2 = -1$ in (26). The β matrix structure in front of V_c is important since its omission leads to a similar expression for the Thomas term, but with the opposite sign. It should be no surprise to find that the relativistic Hamiltonian contains β matrix structures as part of the scalar potential term, by analogy with the mass terms. A Bethe–Salpeter approach starting with a scalar interaction kernel for the $q\bar{q}$ system generates just this β-matrix structure [1].

This demonstration now leads to the Dirac matrix structure of a three-body relativistic scalar potential. The form for a baryon, which again can arise formally out of a Bethe–Salpeter treatment [1], is $\beta_1\beta_2\beta_3 V$ where the β matrices now denote each of the three quarks, and V is the potential given in (6)–(9). The full Hamiltonian is now

$$H_B = \sum_{i=1}^{3}(\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + \beta_i m) - \sum_{i>j=1}^{3}\frac{2\alpha_s}{3}\left[\frac{1}{r_{ij}} - \frac{1}{2r_{ij}}\left(\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_j + \frac{(\boldsymbol{\alpha}_i\cdot\boldsymbol{r}_{ij})(\boldsymbol{\alpha}_j\cdot\boldsymbol{r}_{ij})}{r_{ij}^2}\right)\right] + \beta_1\beta_2\beta_3 V \tag{29}$$

The interaction term proportional to α_s, the strong coupling constant, can be generated from a vector interaction kernel in the Bethe–Salpeter equation, derived in the ladder approximation (and in the instantaneous limit) from one gluon exchange graphs [1]. It is the starting point for a series of FW transformations which can generate the Breit–Fermi relativistic corrections. This procedure is described at some length for the analogous one photon exchange case in QED in [18].

The first FW transformation is performed using, [15]:

$$S = \sum_{i=1}^{3}\frac{1}{2m}\beta_i\boldsymbol{\alpha}_i\cdot\boldsymbol{p}_i + \frac{1}{4}\sum_{i,j=1}^{3}\frac{1}{2m}\beta_i\frac{\alpha_s}{3r_{ij}}\left(\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_j + \frac{(\boldsymbol{\alpha}_i\cdot\boldsymbol{r}_{ij})(\boldsymbol{\alpha}_j\cdot\boldsymbol{r}_{ij})}{r_{ij}^2}\right) \tag{30}$$

thus eliminating odd terms in order m^0. For a complete non-relativistic Hamiltonian plus corrections to order m^{-2} it is necessary to perform two more transformations with more complicated expressions for S. However, it is possible to recognise the order m^{-2} spin–orbit terms due to V after only the first transformation. They are even terms contained in $\frac{1}{2}[S,[S,H_B]]$, using (23), (29) and (30). The Thomas term is

$$H_T = \frac{1}{2}\left[\sum_i \frac{1}{2m}\beta_i \boldsymbol{\alpha}_i.\boldsymbol{p}_i, \left[\sum_j \frac{1}{2m}\beta_j \boldsymbol{\alpha}_j.\boldsymbol{p}_j, \beta_1\beta_2\beta_3 V\right]\right] \tag{31}$$

which can be written as

$$H_T = -\frac{1}{8m^2}\beta_1\beta_2\beta_3 \sum_{i=1}^{3}\{\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i, \{\boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i, V\}\} \tag{32}$$

$$= -\frac{1}{4m^2}\sum_i \boldsymbol{\sigma}_i \cdot (\nabla_i V \times \boldsymbol{p}_i) \tag{33}$$

when β_i is replaced by unity, where $\nabla_i V$ is the derivative of V with respect to quark coordinate $\boldsymbol{r}_i$. The quark coordinates and momenta can be replaced by $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$ and the conjugate momenta $\boldsymbol{p}_\lambda$, $\boldsymbol{p}_\rho$ using (5). Using the centre-of-momentum frame:

$$H_T = -\frac{1}{4m^2}\epsilon^{lkn}\left[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)^n\left(\frac{1}{6}\frac{\partial V}{\partial \lambda^l}p_\lambda^k + \frac{1}{2}\frac{\partial V}{\partial \rho^l}p_\rho^k\right)\right.$$
$$\left.+(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)^n\frac{1}{\sqrt{12}}\left(\frac{\partial V}{\partial \lambda^l}p_\rho^k + \frac{\partial V}{\partial \rho^l}p_\lambda^k\right) + \sigma_3^n\frac{2}{3}\frac{\partial V}{\partial \lambda^l}p_\lambda^k\right] \tag{34}$$

where $l,k,n = 1,3$ label the various vector components.

The expression for H_T is familiar from other derivations of the spin–orbit interaction in baryons, and initially, it might be considered that it could have been written down straight away. However, as far as the author is aware, previous derivations have relied upon V being a sum of two-body terms [3, 4, 8]. The line of development carried through here was chosen since this assumption is not needed: V can be a three-body potential. Since we have this generality, the derivation has been described at some length, with a number of examples.

Now it only remains to evaluate the derivatives of V with respect to $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$. These can be conveniently written as

$$\nabla_\rho V = A\boldsymbol{\rho} + B\boldsymbol{\lambda} \tag{35}$$

and

$$\nabla_\lambda V = B\boldsymbol{\rho} + C\boldsymbol{\lambda} \tag{36}$$

with A, B and C given, with reference to figure 1 and (6)–(9), by

$$A_i = K\left(\frac{\sqrt{2}}{\rho} + \frac{1}{2I_-}\right) \qquad B_i = -\frac{\sqrt{3}K}{2I_-} \qquad C_i = \frac{3K}{2I_-}$$

$$A_{\rm ii} = K\left(\frac{\sqrt{2}}{\rho} + \frac{1}{2I_+}\right) \qquad B_{\rm ii} = \frac{\sqrt{3}K}{2I_+} \qquad C_{\rm ii} = \frac{3K}{2I_+}$$

$$A_{\rm iii} = K\left(\frac{1}{2I_+} + \frac{1}{2I_-}\right) \qquad B_{\rm iii} = \sqrt{3}K\left(\frac{1}{2I_+} - \frac{1}{2I_-}\right) \qquad C_{\rm iii} = 3K\left(\frac{1}{2I_+} + \frac{1}{2I_-}\right) \tag{37}$$

$$A_{\rm iv} = \frac{3K\,(1+\lambda^2/J)}{2\left[\frac{3}{2}\,(\rho^2+\lambda^2)+3J\right]^{1/2}} \qquad B_{\rm iv} = \frac{-3K(\boldsymbol{\lambda}\cdot\boldsymbol{\rho})}{2J\left[\frac{3}{2}\,(\rho^2+\lambda^2)+3J\right]^{1/2}}$$

$$C_{\rm iv} = \frac{3K\,(1+\rho^2/J)}{2\left[\frac{3}{2}\,(\rho^2+\lambda^2)+3J\right]^{1/2}}$$

where

$$I_+ = \left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 + \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2} \qquad I_- = \left(\tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\rho^2 - \sqrt{3}\boldsymbol{\lambda}\cdot\boldsymbol{\rho}\right)^{1/2}$$

and

$$J = \left[\lambda^2\rho^2 - (\boldsymbol{\lambda}\cdot\boldsymbol{\rho})^2\right]^{1/2}. \tag{38}$$

3. Quark oscillator wavefunctions

The spin–orbit term in the three-quark Hamiltonian given by (34) is next examined using quark oscillator wavefunctions describing the P-wave non-strange baryons. These are

$$\begin{aligned} {}_4 8 \quad &: \quad \tfrac{1}{\sqrt{2}}(aaS + bbS) \\ {}_2 10 \quad &: \quad \tfrac{1}{\sqrt{2}}(aSa + bSb) \\ {}_2 8 \quad &: \quad \tfrac{1}{2}(bab + bba + abb - aaa) \end{aligned} \tag{39}$$

in the notation given in [4]. The states are labelled by SU(3) multiplet and total quark spin multiplicity. The components are arranged in space function, SU(3) function, spin function order, and the labels refer to permutation symmetry. S is fully symmetric and a is symmetric (b is antisymmetric) under labels 1, 2 interchange. The spatial functions are

$$\begin{aligned} a \quad &: \quad 2\sqrt{2/3}\,[(m\omega)^2/\pi]\,\lambda Y_{1m}(\Omega_\lambda)\exp\left(-\tfrac{1}{2}m\omega(\rho^2+\lambda^2)\right) \\ b \quad &: \quad 2\sqrt{2/3}\,[(m\omega)^2/\pi]\,\rho Y_{1m}(\Omega_\rho)\exp\left(-\tfrac{1}{2}m\omega(\rho^2+\lambda^2)\right) \end{aligned} \tag{40}$$

where ω is a parameter in the oscillator Hamiltonian $H_{\rm qo}$ for which these are $L = 1$ eigenfunctions:

$$H_{\rm qo} = \frac{1}{2m}\sum_{i=1}^{3}|\boldsymbol{p}_i|^2 + \sum_{i>j}\tfrac{1}{6}m\omega^2 r_{ij}^2 \quad . \tag{41}$$

The spin functions and SU(3) functions are standard: the quartet states are

$$S = \uparrow\uparrow\uparrow \qquad \text{etc} \tag{42}$$

and the doublets

$$a\left(m_s = \tfrac{1}{2}\right) = \frac{1}{\sqrt{6}}\left(2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow\right) \qquad b\left(m_s = \tfrac{1}{2}\right) = \frac{1}{\sqrt{2}}\left(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\right) \tag{43}$$

for example.

States of definite total angular momentum (e.g. ${}_4 8_{5/2}$) are constructed from the expressions in (39) using suitable Clebsch–Gordan coefficients. The matrix elements of H_T can then be computed using (35)–(37). This is a rather lengthy task. The results are given below, incorporating the spin–orbit interactions due to one gluon exchange (i.e. for the total spin–orbit Hamiltonian H_{so}).

$$\begin{aligned}
&< {}_4 8_{5/2} | H_{so} | {}_4 8_{5/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\,(m\omega\alpha_s - KI) \\
&< {}_4 8_{3/2} | H_{so} | {}_4 8_{3/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(-\tfrac{2}{3}m\omega\alpha_s + \tfrac{2}{3}KI\right) \\
&< {}_4 8_{3/2} | H_{so} | {}_2 8_{3/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(-\tfrac{\sqrt{10}}{9}m\omega\alpha_s + \tfrac{\sqrt{10}}{3}KI\right) \\
&< {}_2 8_{3/2} | H_{so} | {}_2 8_{3/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(\tfrac{1}{3}m\omega\alpha_s - \tfrac{1}{3}KI\right) \\
&< {}_4 8_{1/2} | H_{so} | {}_4 8_{1/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(-\tfrac{5}{3}m\omega\alpha_s + \tfrac{5}{3}KI\right) \\
&< {}_4 8_{1/2} | H_{so} | {}_2 8_{1/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(\tfrac{2}{9}m\omega\alpha_s - \tfrac{2}{3}KI\right) \\
&< {}_2 8_{1/2} | H_{so} | {}_2 8_{1/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(-\tfrac{2}{3}m\omega\alpha_s + \tfrac{2}{3}KI\right) \\
&< {}_2 10_{3/2} | H_{so} | {}_2 10_{3/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(\tfrac{1}{9}m\omega\alpha_s + \tfrac{1}{3}KI\right) \\
&< {}_2 10_{1/2} | H_{so} | {}_2 10_{1/2} > = \sqrt{\frac{\omega}{2\pi m^3}}\left(-\tfrac{2}{9}m\omega\alpha_s - \tfrac{2}{3}KI\right)
\end{aligned} \tag{44}$$

where

$$KI = \frac{(m\omega)^{7/2}}{6\sqrt{2}\ \pi^{5/2}} \int \mathrm{d}^3\boldsymbol{\rho}\ \mathrm{d}^3\boldsymbol{\lambda}\left(A\rho^2 + 2B\boldsymbol{\lambda}\cdot\boldsymbol{\rho} + C\lambda^2\right)\exp\left(-m\omega(\rho^2 + \lambda^2)\right) \quad . \tag{45}$$

There are some interesting points to note in these results. Firstly, all the contributions due to the three-quark flux-tube potential are given in terms of the single integral I. In this expression, the quantities A, B and C take on different functional forms (listed in (37)) depending on the relative orientation of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, using the criteria (10)–(12). Equation (45) hides many difficulties involving complicated boundary conditions. Some of the integrations can be done analytically, however, leaving those over the quantities ρ, λ and α, the angle between $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$. The remaining integral can be done numerically by Monte Carlo selection over the integration space, weighted according to the exponential in the integrand. It is during this calculation that the complicated boundary conditions are taken into account, by simple conditional checks as to which values of A, B, and C are required for a particular selected point. The complexity of the boundary conditions probably precludes an analytical calculation. The result of the computation is $I = 0.542(6)$.

The important comparison we wish to make is with the spin–orbit contributions due to the two-body scalar potential V_{2b} in (13). Taking into account the permutation symmetry of the wavefunctions, the effects can be represented by the choice, in the above formalism, of $B = C = 0$ and

$$A = \frac{3\sqrt{2}fK}{\rho} \quad . \tag{46}$$

In this case, the full integral in (45) can be done analytically, giving $I = f$. As mentioned before, a value $f = 0.5493$ was taken in [8]. This was chosen in order to minimise the expectation value of the residual three body potential V_{3b} within the ground state baryon wavefunction, but clearly the choice also turns out to provide quite an accurate approximation to the spin–orbit splittings due to the three-quark potential V. The Thomas term due to V_{3b} was ignored in [8]: in fact our work shows that its effect within that scheme is negligible.

It is worth briefly comparing the above result for the three-body potential with the general consideration of spin–orbit splittings due to a scalar potential given by Gromes [11], which concludes that two quantities F and G parametrise the matrix elements. G is zero for two-body potentials: unfortunately it is zero also for V, since it is a local potential. The value of F can be easily found to be equal to $-\frac{4}{3}(\omega/2\pi m^3)^{1/2}KI$.

4. Conclusions

We have examined the spin–orbit interaction between the three quarks in a baryon, due to a realistic three-body potential based on the flux-tube model. This interaction, which is purely a Thomas precession effect since we take the flux-tube potential to be scalar in character, is found by using a Foldy–Wouthuysen transformation on the relativistic three-quark Hamiltonian. The procedure is a development of the two-particle case considered by Chraplyvy [15] and has the advantage that a three-body potential can be treated. The resulting spin–orbit potential is given in (34).

Previously, the spin–orbit interaction in baryons was derived from a combination of two-quark potentials, (13), which were an approximation to the more realistic three-body confining potential given in (6)–(9). This was done in order to simplify the derivation. As the spin–orbit interaction depends on derivatives of the non-relativistic potential (e.g. (28)), this approach seems at first sight to be suspect. Since a comparison of the theoretical spin–orbit splittings in baryons with experiment still presents some

problems even in very sophisticated models such as [8], we have compared the two-body and three-body approaches using harmonic oscillator wavefunctions of the P-wave non-strange baryons. The results, together with the effects of vector gluon exchange, are summarised in (44).

Similar patterns of relative splittings in the two cases emerge. The difference between the two- and three-body approaches lies in the value of an integral I ((45)), which involves derivatives of the non-relativistic confining potential, as expected. For the three-body case, $I = 0.542(6)$. For the two-body approach of [8], $I = f$ where f is a constant appearing in the approximate two-body potential, (13). In order to obtain a good approximation to the non-relativistic baryon energies, f was chosen in [8] to be equal to 0.5493. This paper has shown that this choice also results in a good approximation to the spin–orbit contributions to the baryon masses.

To conclude, it appears that a more careful consideration of the Thomas effect in a baryon system yields a result little different to the conclusions of previous, more approximate approaches.

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