A Very Short Introduction to Dynamic Optimisation

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1 Introduction

Economics is often interested in the behaviour of individuals or agents. In the models we will study, these agents are assumed to behave rationally, that is, taking decisions that optimise their utility (in the case of households) or profits (in the case of firms). Optimisation implies that agents maximise their utility/profits subject to the restrictions they face. When this optimisation process spans more than one period, we call it Dynamic Optimisation. In this short tutorial, we will introduce some tools that allow us to analyse the agent's behaviour.

2 Dynamic Optimisation

We will analyse the case of an economic agent (we will focus on a household) who optimises a concave [1] utility function U which depends (positively) on consumption, c_t and (negatively) on the number of hours worked, n_t: $U(c_t, n_t)$.[2] We assume that the agent lives from period 0 to period T, so she will take decisions on how much consume or work in each period t during her lifetime. However, the agent discounts future utility by a factor $0 < \beta < 1$ so that her present-value utility becomes:

$$V_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, n_t) \tag{1}$$

The agent faces a budget constraint (holding $\forall t = 0 \dots \infty$) which imposes a trade-off between consuming or working:

$$c_t + k_{t+1} \leq w_t n_t + (1 + r_t)k_t \tag{2}$$

where k_t are savings that allow the agent to move consumption over time, w_t is the real wage, and r_t an interest rate paid over the savings.

We will make two further observations:

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[1] Do you know why concavity of this function is relevant?

[2] Alternatively, we could also say that the individual obtains positive utility from consumption and leisure (l_t): $U(c_t, l_t)$. If we normalise the available time during the day to 1, there is a direct relationship between the time we spend consuming leisure and the time spend working: $n_t = 1 - l_t$ (i.e. taking a decision on n_t implies a decision on l_t). We can then write $U(c_t, 1 - n_t)$. Notice that if want to take the derivative of this expression with respect we can apply the chain rule: $\frac{\partial U(c_t, 1-n_t)}{\partial n_t} = -U_l(c_t, l_t)$.

1. In the optimum, the budget constraint holds with equality: $c_t + k_{t+1} = w_t n_t + (1+r_t)k_t$, that is, no resources are left un-used. Otherwise, we would be facing an optimisation problem with inequality constraints which would require the use of the *Kuhn-Tucker* conditions.

2. We assume that there is no uncertainty, so future values of the variables (e.g. c_{t+1}, w_{t+1}, k_{t+2}...) for $t+1$ to ∞ are known. The alternative would be to recognise the presence of uncertainty and set up a stochastic dynamic optimisation problem, which would involve expectations over unknown objects.

2.1 The Method of Lagrange Multipliers

To maximise Equation 1 subject to Equation 2 we use the method of *Lagrange Multipliers*. This procedure involves setting up the *Lagrangian* function

$$\mathcal{L}(c,n,k,\lambda) = V_0 + \sum_{t=0}^{\infty} \lambda_t \left[w_t n_t + (1+r_t)k_t - c_t - k_{t+1} \right] \tag{3}$$

where λ_t are called the *Lagrange Multipliers*. In our case: [3]

$$\mathcal{L} = \sum_{t=0}^{T} \{\beta^t U(c_t, n_t) + \lambda_t \left[w_t n_t + (1+r_t)k_t - c_t - k_{t+1} \right]\} \tag{5}$$

The first order conditions for $t = 0 \ldots \infty$ are obtained by taking derivatives with respect to the arguments:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t U_c - \lambda_t = 0 && (6)\\
\frac{\partial \mathcal{L}}{\partial n_t} &= \beta^t U_n + \lambda_t w_t = 0 && (7)\\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= -\lambda_t + \lambda_{t+1}(1+r_{t+1}) = 0 && (8)\\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= w_t n_t + (1+r_t)k_t - c_t - k_{t+1} = 0 && (9)
\end{aligned}$$

where $U_{c,t} = \frac{\partial U(c_t,n_t)}{\partial c_t}$ and $U_{n,t} = \frac{\partial U(c_t,n_t)}{\partial n_t}$. Note that the above conditions hold for $t = 0 \ldots \infty$, so in each period t, these conditions are the rules that households will use to behave optimally.

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[3] Notice that in this case the Lagrange Multipliers are implicitly including a time discount factor. We can make this explicit and then write:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \{U(c_t, n_t) + \lambda_t \left[w_t n_t + (1+r_t)k_t - c_t - k_{t+1} \right]\} \tag{4}$$

The optimality conditions resulting from this problem will have the same interpretation.

2.2 Interpretation of First Order Conditions

Combining Equations 6 and 7 we obtain an intratemporal restriction relating consumption and leisure decisions:

$$\frac{U_{n,t}}{U_{c,t}} = -w_t \tag{10}$$

Particularly, the left-hand side of Equation 10 is the Marginal Rate of Substitution of consumption and hours worked (the slope of the indifference curve), which is equated to the ratio of prices of hours worked (w_t) and consumption (normalised to 1). More importantly, Equation 10 defines the labour supply of the agent.

On the other hand, by combining Equation 8 in period t and $t+1$ we obtain the Euler Equation which relates the utility of consumption today and consumption tomorrow:

$$U_{c,t} = \beta U_{c,t+1}(1 + r_{t+1}) \tag{11}$$

This condition tells us how the agent should allocate consumption intertemporally. Particularly, it says that the marginal utility of consumption today must be equalised to the discounted future marginal utility of consumption tomorrow; note that when the agent declines consuming one unit of the good today, she obtains $(1+r_{t+1})$ to be consumed tomorrow.

3 Readings

A more extensive analysis of dynamic optimisation can be found in the appendixes of the following books:

- A. Mas-Colell, M.D. Whinston and J.R. Green: Microeconomic Theory, *Oxford University Press* (1995).

- M. Wickens: Macroeconomic theory: A dynamic General Equilibrium Approach, *Princeton University Press* (2009)